Selected solutions PS 9

1. $$\left[\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right] u(\lambda x, \lambda^2 t)$$

$$= \frac{\partial}{\partial t}\left[u(\lambda x, \lambda^2 t)\right] - k\frac{\partial^2}{\partial x^2}\left[u(\lambda x, \lambda^2 t)\right]$$

$$= \lambda^2 \cdot u_t(\lambda x, \lambda^2 t) - k \cdot \lambda^2 u_{xx}(\lambda x, \lambda^2 t)$$

$$= \lambda^2 \cdot \left[u_t(\lambda x, \lambda^2 t) - k u_{xx}(\lambda x, \lambda^2 t)\right]$$

$$= \lambda^2 \cdot 0$$ $\Leftarrow$ using u satisfies heat equation $u_t - k u_{xx} = 0$

$$= 0$$

So $u(\lambda x, \lambda^2 t)$ satisfies Heat Equation

On the other hand, $u(\lambda x, -\lambda^2 t)$ does not satisfies the Heat Equations

4. We have deduced that the general solutions are

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{\pi^2 n^2}{L^2} kt} \sin\frac{n\pi x}{L}$$ with $L = 1$ in this question

Thus $u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\pi^2 n^2 kt} \sin(n\pi x)$.

The initial condition gives $-6\sin(6\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$

Using the orthogonalities of $\sin(n\pi x)$, we "observe" that $a_n = 0$ except for $n = 6$

And $a_6 = -6$. (the $n=6$ term need to match)

So the solution to this Dirichlet problem is

$$u(x,t) = -6 e^{-36\pi^2 k t} \sin(6\pi x)$$

5. Using separation of variables, suppose

$$u(x,t) = X(x)T(t).$$

plug into equation get

$$X\dot{T} = k X'' T$$

$$\frac{\dot{T}}{kT} = \frac{X''}{X} = -\lambda \quad \text{a constant.}$$

We get 2 ODEs

$$\begin{cases} X'' = -\lambda X \\ \dot{T} = -\lambda k T \end{cases}$$

The 1st ODE and the boundary condition gives an eigenvalue problem.

$$\begin{cases} X'' = -\lambda X \\ X'(0) = 0, \; X'(L) = 0 \end{cases}$$

Using integration by parts, we have shown in problem set 5 Question 2 the same eigenvalue problem that $\underline{\lambda \geq 0}$.

So $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

and $X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$

The first bondary condition gives

$$0 = X'(0) = B\sqrt{\lambda} \quad , \text{ so } B = 0 \text{ and } A \neq 0$$

the second bondary condition gives

$$0 = X'(L) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}\cdot L)$$

we have $\sqrt{\lambda}\cdot L = n\pi$, $\quad n = 1, 2, \cdots$

So the eigenvalues are $\lambda_n = \frac{n^2\pi^2}{L^2}$

The eigenfunctions are

$$X_n(x) = \cos\frac{n\pi x}{L}$$

knowing λ_n, we solve $\dot{T} = -k\lambda_n T$ and get

$$T_n(t) = e^{-\frac{n^2\pi^2}{L^2}kt}$$

For $n=0$, i.e. $\lambda = 0$, we get

$$X'' = 0, \quad X(x) = a_0 + b_0 t$$

the bondary condition $X'(0) = X'(L) = 0$ gives $b_0 = 0$

so $X_0(x) = \text{Const}$, solving for $\dot{T} = 0$ we get

$$T_0(t) = \text{Const}$$

The general solutions are

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n X_n(x) T_n(t)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{L^2}kt} \cos\frac{n\pi x}{L} .$$

6. (1) plug in $t=0$, we get

$$1 = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

using that $\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n \neq m \\ \frac{L}{2}, & n = m \end{cases}$ (*)

we have (by multiply both sides by $\sin\frac{m\pi x}{L}$)

$$\int_0^L \cos\frac{m\pi x}{L} = \sum_{n=1}^{\infty} a_n \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = a_m \cdot \frac{L}{2}$$

we get $\quad a_m = \frac{2}{L}\int_0^L \cos\frac{m\pi x}{L}$

$$= \frac{2}{L}\frac{L}{m\pi}\left[\sin\frac{m\pi x}{L}\right]\Big|_0^L$$

$$= \frac{2}{L}\frac{L}{m\pi}[0-0]$$

$$= 0 \qquad \text{for } m \geq 1$$

so the initial condition gives $1 = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} = a_0$

The solution is

$$u(x,t) \equiv 1$$

(2) Notice $\cos^2\left(\frac{\pi x}{L}\right) = \dfrac{\cos\frac{2\pi x}{L}+1}{2}$

so $\frac{1}{2} + \frac{1}{2}\cos\frac{2\pi x}{L} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$

we can use ther orthogonality condition (**)

and "observe" that

$$a_0 = \frac{1}{2}, \quad a_2 = \frac{1}{2}, \quad a_n = 0 \text{ for } n \neq 0, 2$$

so plug into the general solutions obtained in 5 and get

$$u(x,t) = \frac{1}{2} + \frac{1}{2}e^{-\frac{4\pi^2}{L^2}kt}\cos\frac{2\pi x}{L}$$