

Week 10: PDE revision and the wave equation

Finite differencing, pseudospectral methods and their application to the heat equation

Dr K Clough, Topics in Scientific computing, Autumn term 2023

Plan for today

1. Revision of PDE types and their properties
2. Problems with PDEs - well posedness
3. Problems with PDEs - Von Neumann stability and the CFL condition
4. Solving second order in time PDEs - solution of the wave equation

Classification of second order PDEs

Consider the most general second order PDE for 1 dependent variable with 2 independent variables:

$$A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F = 0$$

The equation is classified based on the discriminant $\Delta = B^2 - 4AC$:

$\Delta < 0$ Elliptic

$\Delta = 0$ Parabolic

$\Delta > 0$ Hyperbolic

Example 1: The heat equation

The heat equation, (α is a positive constant, S is any function of u , x and t but not their derivatives)

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + S$$

What type is this equation?

Example 1: The heat equation

The heat equation is a parabolic equation $\Delta = 0$

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + S \quad \rightarrow A = \alpha, E = -1, B = C = D = 0, F = S$$

This equation is ***first order in time***, so solutions will evolve in time as exponentials in response to an instantaneous source. The dependence on the ***second derivative in space*** means that it has a tendency to smooth the solution - any bumps in the solution decrease in time assuming α is positive.

Example 2: The wave equation

The wave equation (c is a positive constant, S is any function of u , x and t but not their derivatives)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + S$$

What type is this equation?

Example 2: The wave equation

The wave equation is a hyperbolic equation $\Delta > 0$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + S \quad \rightarrow A = c, C = -1, B = D = E = 0, F = S$$

This equation is **second order in time**, so solutions will evolve in time with oscillations in response to an instantaneous source. The dependence on the **second derivative in space** means that it has a tendency to pull any bumps back towards zero displacement.

Hyperbolic equations have a finite speed of propagation of information - c .

Example 3: Poisson's equation

The Poisson equation (f is any function of u , x and t but not their derivatives)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$$

What type is this equation?

Example 3: Poisson's equation

The Poisson equation is an elliptic equation $\Delta < 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \quad \rightarrow A = 1, C = 1, B = D = E = 0, F = -f$$

This equation is ***second order in both dimensions, which are usually thought of as two spatial directions*** (for reasons we will discuss next). If the source f is zero it is called Laplace's equation, and for zero boundary conditions the solution is a constant. A non zero source creates a displacement or bump in the solution.

Elliptic equations have an infinite speed of propagation of information.

Example 3: Poisson's equation in 1D

The Poisson equation (f is any function of u , x and t but not their derivatives)

$$\frac{\partial^2 u}{\partial x^2} = f$$

What type is this equation?

Example 4: Poisson's equation in 1D

The Poisson equation (f is any function of u , x and t but not their derivatives)

$$\frac{d^2u}{dx^2} = f$$

Trick question! This is just an ODE like we studied before as there is only one independent variable!

Example 5: Katy's equation

Katy's equation (f is any function of u , x and t but not their derivatives)

$$\frac{\partial^2 u}{\partial t^2} + (t - 10) \frac{\partial^2 u}{\partial y^2} = f$$

What type is this equation?

Example 5: Katy's equation

Katy's equation (f is any function of u , x and t but not their derivatives)

$$\frac{\partial^2 u}{\partial t^2} + (t - 10) \frac{\partial^2 u}{\partial y^2} = f$$

This equation changes character at $t=10$ - before it is hyperbolic and after it is elliptic.

A system of PDEs can be of mixed type (e.g. Navier Stokes is mixed parabolic/hyperbolic) and they can change type at different points in space and time.

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Well posed problems - very active area of QMUL research!

- QMUL Maths is one of the leading places for solving issues of well-posedness.
- e.g. Prof Claudia Garetto of geometry, analysis and gravitation centre



On the well-posedness of weakly hyperbolic equations with time-dependent coefficients

Claudia Garetto¹, Michael Ruzhansky²  


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Abstract

In this paper we analyse the Gevrey well-posedness of the Cauchy problem for weakly hyperbolic equations of general form with time-dependent coefficients. The results involve the order of lower order terms and the number of multiple roots. We also derive the corresponding well-posedness results in the space of Gevrey Beurling ultradistributions.

Well posed problems

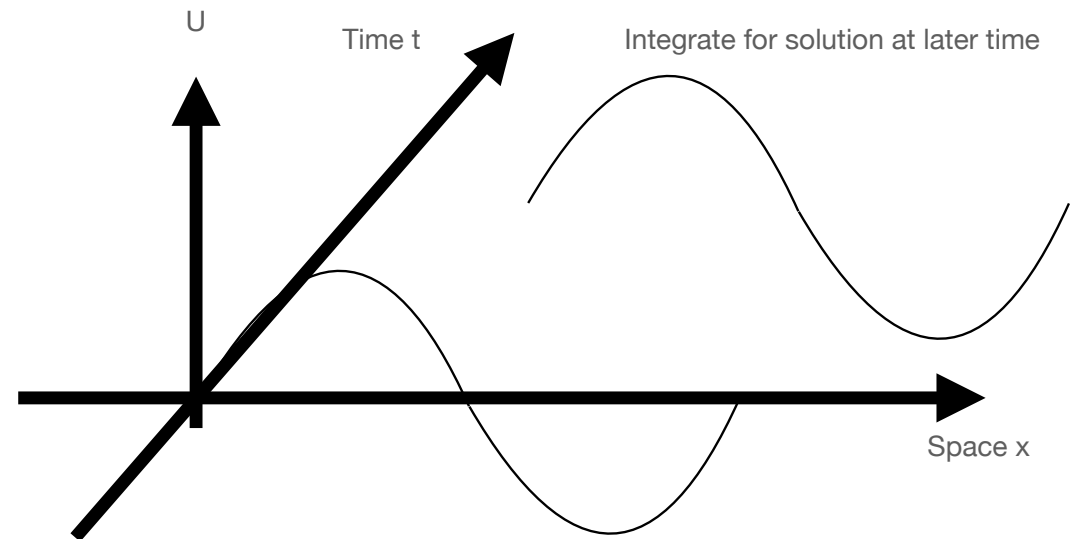
- An initial value / Cauchy problem is well posed if:
 - A solution exists
 - The solution is unique
 - The solution depends continuously on the initial data



What is an initial value problem/Cauchy problem?

Initial value problem

- One of the independent variables is thought of as “time” (doesn’t have to actually **be** time)
- Boundary value is provided as a value of the function at some (arbitrary) time $t=0$
- Full solution is found by integrating in time
- We will see an alternative (boundary value solution via relaxation) next week



Initial condition $u(x, t=0)$ provided for all space at $t=0$

Well posed problems

- An initial value / Cauchy problem is well posed if:
 - A solution exists
 - The solution is unique
 - The solution depends continuously on the initial data

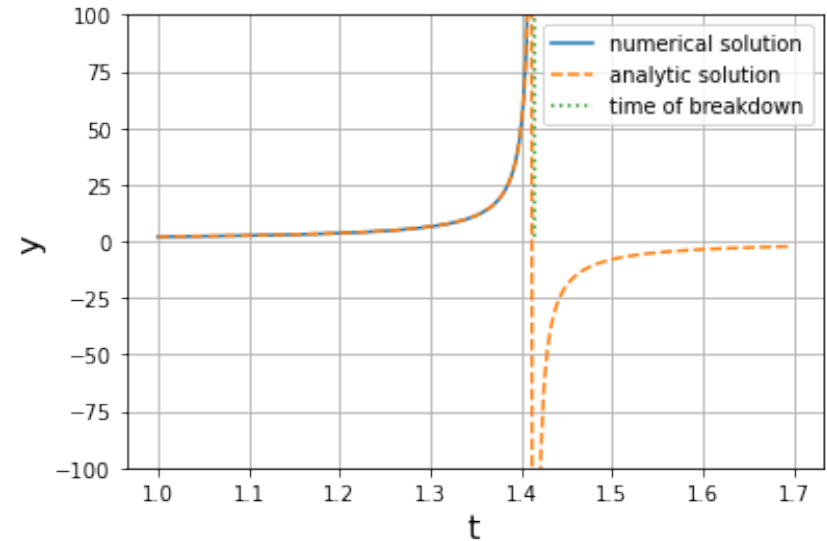


What did it mean for the solution to depend continuously on the initial data?

Well posed problems

Recall for ODEs:

- If $x_1(0) = a, x_2(0) = a + \delta$ it tells us that the solution changes by an amount that is bounded by δe^{Lt} where L is some constant value - this is the meaning of “***depends continuously on the initial data***”.
- We had the example that blows up at a value that depends on the initial conditions, so that a small change results in a change that is not bounded by an exponential



Well posed problems

- An initial value / Cauchy problem is well posed if:
 - A solution exists
 - The solution is unique
 - The solution depends continuously on the initial data



How can a solution not exist?

Well posed problems

For Laplace's equation

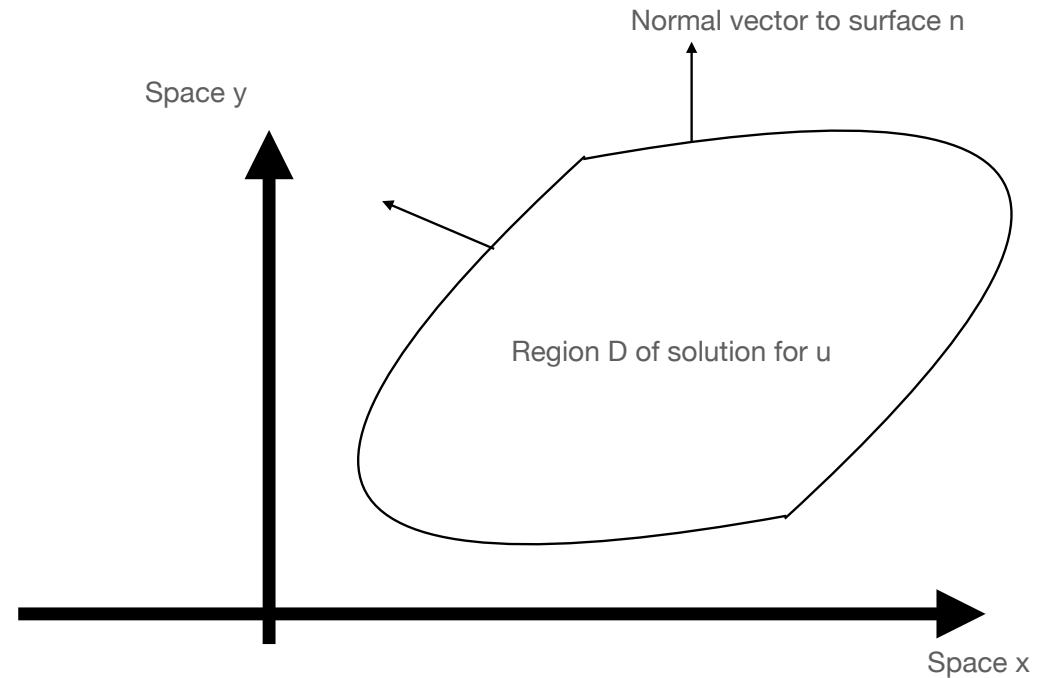
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with fixed boundary conditions

$$\nabla u \cdot n = g(x, y) \quad (x, y) \in \partial D$$

No solution exists if $\int_{\partial D} g(x, y) ds \neq 0$

A nice detailed explanation is here: <https://youtu.be/BmTFbUAOeec?si=22bdWktp55xLcT3s>



Well posed problems

- An initial value / Cauchy problem is well posed if:
 - A solution exists
 - The solution is unique
 - The solution depends continuously on the initial data



How can a solution not be unique?

Well posed problems

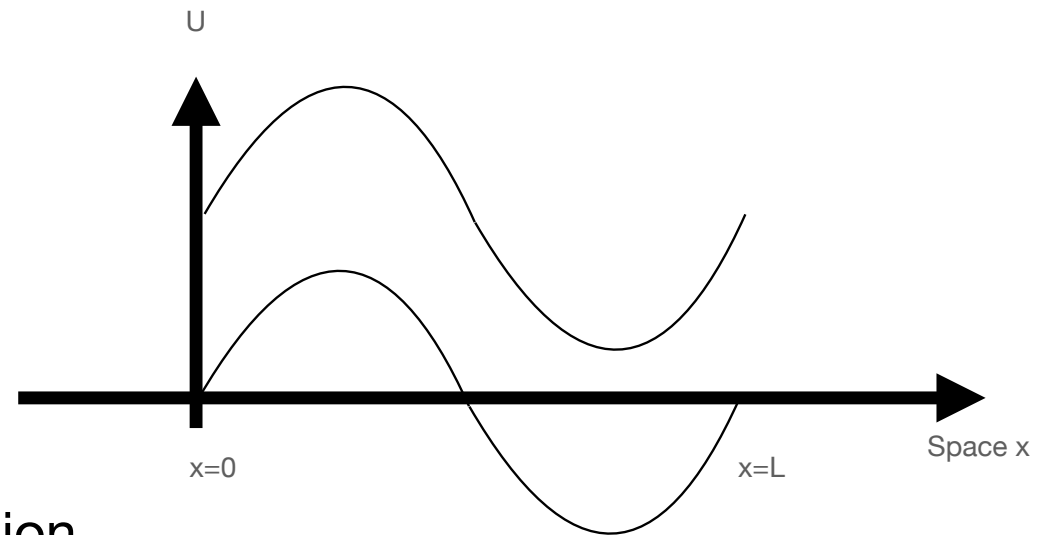
Consider Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$$

with periodic boundary conditions

$$u(x = L) = u(x = 0)$$

Then for any solution $u(x, t)$ the solution $\bar{u}(x, t) = u(x, t) + C$ with C a constant is also a solution.



Well posed problems

- Theorems in mathematics guarantee the (local) well-posedness of linear and quasi-linear* strongly hyperbolic* and parabolic PDEs.
- Elliptic PDEs do not admit a well-posed IVP. This does not (necessarily) mean they cannot be solved, just that another method may be required.
- When in a correct numerical implementation one increases the resolution and the solution blows up faster, that usually implies an ill-posed initial value problem.



*We will discuss the exact meaning of these terms next week. For now just think of hyperbolic and parabolic equations as generally ok.

Well posed problems - why elliptic equations fail as an initial value problem

Consider Laplace's equation but treat one of the directions as a "time":

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

And propose a wave like solution

$$u(x, t) = \exp(i[\omega t - kx])$$

Then

$$-\omega^2 u - k^2 u = 0 \quad \implies \quad \omega = \pm i |k| \quad \implies \quad u(x, t) = A \exp(|k|t + ikx) + \dots$$

Which blows up exponentially at a faster rate for higher k (= shorter wavelengths)

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Von Neumann stability analysis and the CFL condition (Courant Friedrich Lewy)

- Like for ODEs, numerical schemes for PDEs can be unstable, and they have to be analysed for each PDE and PDE scheme separately
- For an initial value problem, this usually results in a “**CFL condition**” on the time step of the form:

$$\begin{aligned}\Delta t &= \lambda \Delta x && \text{for hyperbolic equations} \\ \Delta t &= \lambda \Delta x^2 && \text{for parabolic equations}\end{aligned}$$

- The main method to determine the CFL number λ is called the Von Neumann stability analysis. It is a *necessary but not sufficient* condition for stability.
- Just by physical arguments, for disturbances travelling at a speed of c we should expect $\lambda \leq 1/c$, and our physical/mathematical intuition can always do the job for us - starting with our intuitive stability condition, we could use trial and error to find how high/low λ can be before the code becomes numerically unstable

Von Neumann stability analysis

Method:

1. Assume a harmonic perturbation of the form

$$u(x, t) = \exp(i[\omega t - kx])$$

2. Calculate for the given numerical scheme the amplification factor between timesteps

$$\Lambda = \frac{u_i^{n+1}}{u_i^n}$$

3. Require $|\Lambda| \leq 1$ for the solution to not be amplified, which gives rise to a condition

$$\Delta t = \lambda \Delta x \quad \text{where } \lambda \text{ is the CFL number}$$

Plan for today

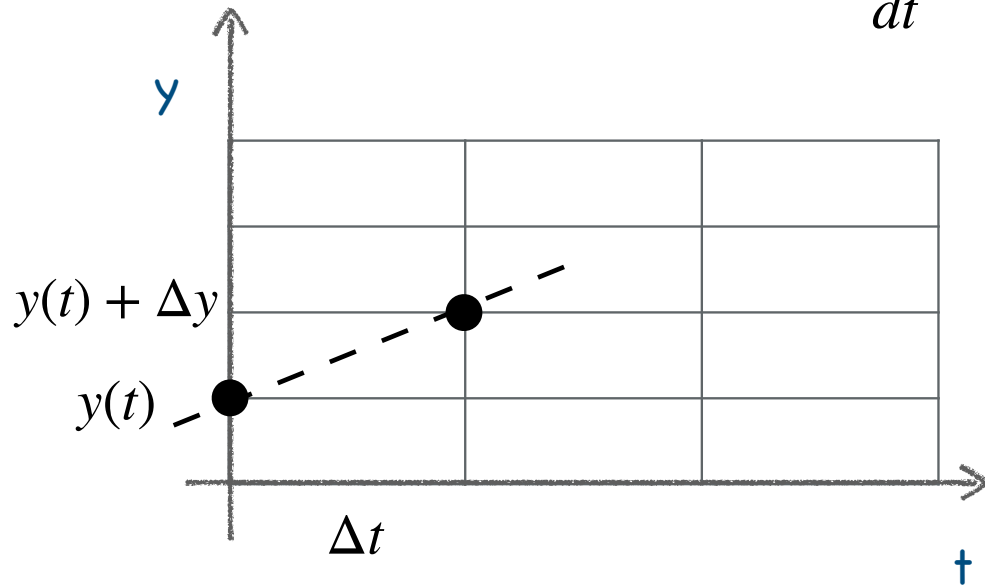
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Recall: How do I integrate second order ODEs numerically?

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + f(y, t) = 0$$

$$\left\{ \begin{array}{l} \frac{dv}{dt} - v + f(y, t) = 0 \\ \frac{dy}{dt} = v \end{array} \right.$$

1. Decompose the second order equation into two first order ones



$$\Delta v = \Delta t (v - f(y, t))$$

$$\Delta y = v \Delta t$$

2. Solve as a dimension 2 first order system

Solving second order PDEs - the wave equation

Consider the wave equation for u :

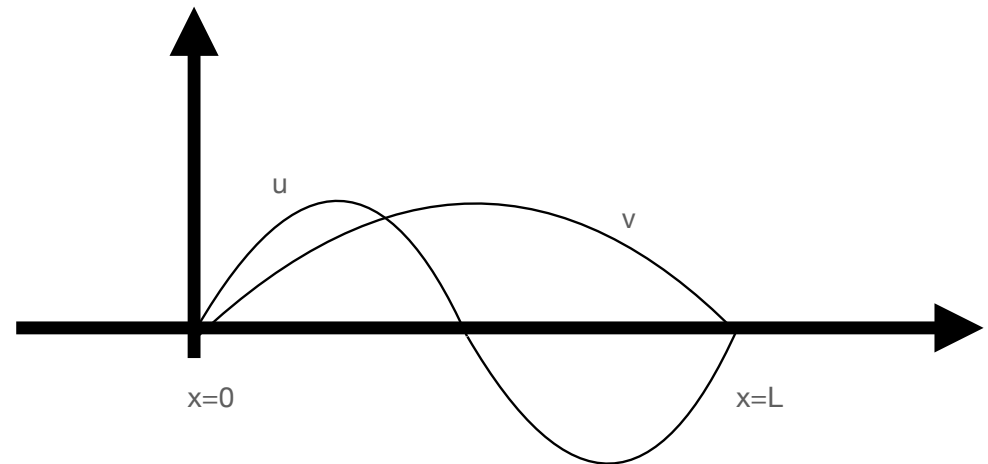
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

And define the time derivative to be

$$v(x, t) = \frac{\partial u}{\partial t}$$

Then we solve the coupled system:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial t} = v$$



Wave equation - matrix representation

Recall that we can also represent this in matrix form: $\frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\frac{\partial u}{\partial t} = v$

dv/dt

2
3
1
-2
-2
-2

=

Matrix D^2

X	x				
1	-2	1			
	1	-2	1		
		1	-2	1	
			1	-2	1
				x	x

•

u

0
1
3
2
1
0

All blank entries zero

Wave equation - matrix representation

Recall that we can also represent this in matrix form: $\frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\frac{\partial u}{\partial t} = v$

du/dt

2
3
1
-2
-2
-2

=

Matrix I

1					
	1				
		1			
			1		
				1	
					1

•

v

0
1
3
2
1
0

All blank entries zero

Wave equation - state vector in python

Need to unpack and repack the state vector in python.

Some useful commands:

```
u0 = get_y_test_function(x_values)
v0 = np.zeros_like(u0)
y0 = np.concatenate([u0, v0])
```

```
# Just for readability
[u, v] = np.array_split(current_state, 2)
dydt = np.zeros_like(current_state)
dudt, dvdt = np.array_split(dydt, 2)
dudt[:] = v
```

y {

u

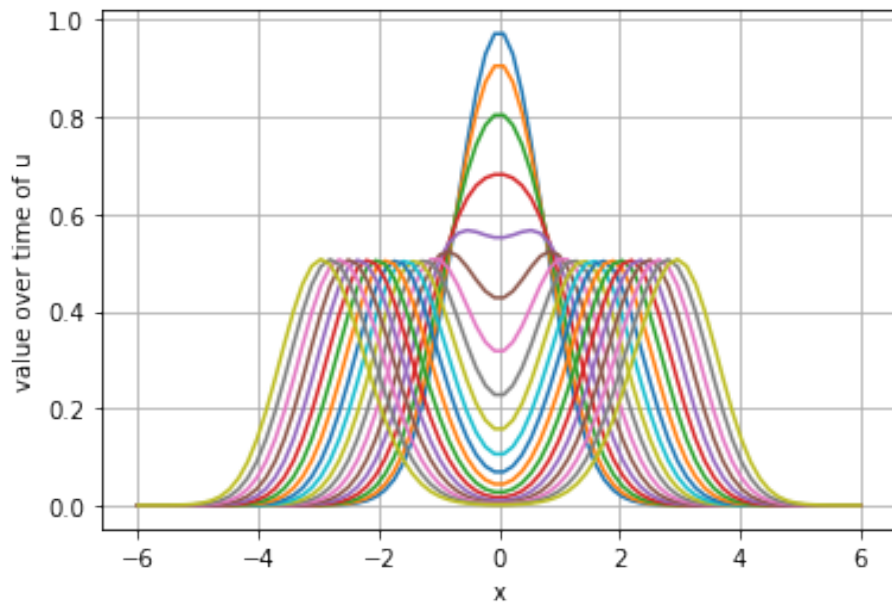
2
3
1
-2
-2
-2

v

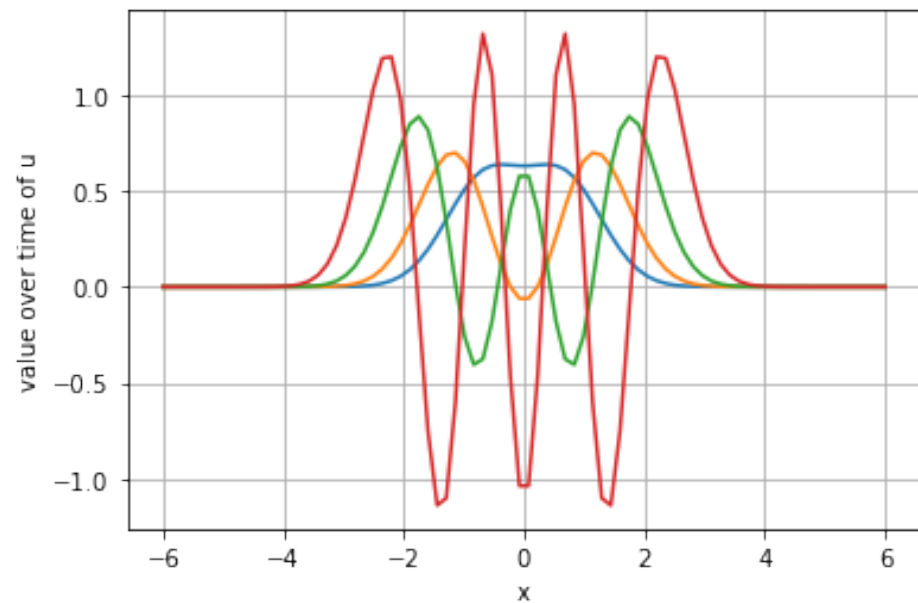
0
1
3
2
1
0

Wave equation - tutorial

In the tutorial you will update the heat equation code from last week for the wave equation, and test the CFL condition.



CFL condition respected



CFL condition not respected

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