MTH6107 Chaos & Fractals

Solutions 6

EXAM QUESTIONS: Exercises 1–5 below correspond to the various parts of Question 4 on the January 2023 exam paper, and Exercise 6 corresponds to Question 1 on the same exam paper.

Exercise 1. For the function $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \sum_{i=0}^{9} x^{2i+1}$, give a formula for the derivative $f_1'(x)$.

The derivative is given by

$$f_1'(x) = \sum_{i=0}^{9} (2i+1)x^{2i} = 1 + \sum_{i=1}^{9} (2i+1)x^{2i} .$$

Exercise 2. Using properties of the derivative f_1', or otherwise, show that the only periodic point for f_1 is the fixed point at 0.

Since each $x^{2i} \geq 0$ for all $x \in \mathbb{R}$, from the formula for f_1' we see that $f_1'(x) \geq 1 > 0$ for all $x \in \mathbb{R}$.

Clearly 0 is a fixed point.

To see that there are no other fixed points, note that the fixed point equation $f_1(x) = x$ becomes $\sum_{i=1}^{9} x^{2i+1} = 0$, but 0 is the only solution to this because $\sum_{i=1}^{9} x^{2i+1}$ is a strictly monotone function of x.

To see that there are no points of period strictly larger than 1 it suffices to note that f_1 is a diffeomorphism, and is orientation-preserving since $f_1' > 0$, and then cite the result proved in lectures that orientation-preserving diffeomorphisms do not have periodic points of period strictly larger than 1.

Exercise 3. For the function $f_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_2(x) = \begin{cases} -2(1+x) & \text{for } x < 0 \\ x - 2 & \text{for } x \geq 0 \,, \end{cases}$$

evaluate the set $\{n \in \mathbb{N} : f_2 \text{ has a point of least period } n\}$, being careful to justify your answer.

The map f_2 is continuous, and has an orbit of least period 3, namely $\{-2, 2, 0\}$, therefore by Sharkovskii's Theorem, $\{n \in \mathbb{N} : f_2 \text{ has a point of least period } n\}$ is the whole of $\mathbb{N}$.

Exercise 4. For the function $f_3 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_3(x) = \begin{cases} -2(1+x) & \text{for } x < 0 \\ x/2 - 2 & \text{for } x \geq 0 \,, \end{cases}$$

evaluate the set $\{n \in \mathbb{N} : f_3 \text{ has a point of least period } n\}$, being careful to justify your answer.

We claim that $\{n \in \mathbb{N} : f_3 \text{ has a point of least period } n\} = \{1, 2, 4\}$.

To see this, note that the map f_3 is continuous, and has an orbit of least period 4, for example $\{-2, 2, -1, 0\}$, therefore by Sharkovskii's Theorem, $\{n \in \mathbb{N} : f_3 \text{ has a point of least period } n\}$ contains $\{1, 2, 4\}$. (Alternatively, computation shows that $-2/3$ is the unique fixed point, and $\{-3/2, 1\}$ is the unique 2-cycle).

We now justify the assertion that if $n \notin \{1, 2, 4\}$ then f_3 does not have an n-cycle.

First note that if $x < -2$ then $f_3(x) > 2$, and if $y > 2$ then $f_3^n(y) \in [-2, 2]$ for some $n \in \mathbb{N}$, and if $z \in [-2, 2]$ then $f_3(z) \in [-2, 2]$, therefore all periodic points of f_3 belong to $[-2, 2]$.

First we claim that every point in

$$X := [-2, 2] \setminus \left(\{-\frac{3}{2}, 1\} \cup (-1, 0) \right) = [-2, -\frac{3}{2}) \cup (-\frac{3}{2}, -1] \cup [0, 1) \cup (1, 2]$$

has least period 4. To see this note that $f_3((1, 2]) = (-3/2, -1]$, $f_3((-3/2, -1]) = [0, 1)$, $f_3([0, 1)) = [-2, -3/2)$, and $f_3([-2, -3/2)) = (1, 2]$, and if $x \in (1, 2]$ then $f_3(x) = x/2 - 2$, $f_3^2(x) = -2(1 + x/2 - 2) = 2 - x$, $f_3^3(x) = (2 - x)/2 - 2 = -1 - x/2$, $f_3^4(x) = -2(1 - 1 - x/2) = x$.

Next we note that if $x \in (-1, 0)$ is not the fixed point $-2/3$ then x is eventually periodic: either $f_3^n(x) \in (1, 2]$ for some $n \in \mathbb{N}$ so x is eventually periodic of least period 4, or $f_3^n(x) = -3/2$ for some $n \in \mathbb{N}$ so x is eventually periodic of least period 2 (there are countably many such points: $\ldots - 23/32 \mapsto -9/16 \mapsto -7/8 \mapsto -1/4 \mapsto -3/2$).

Therefore we have shown that there are no periodic points whose least period is not equal to either 1, 2 or 4.

Exercise 5. Without using Sharkovskii's Theorem, show that every continuous function $f : \mathbb{R} \to \mathbb{R}$ which has a periodic orbit must have a fixed point. [*Hint: Use the Intermediate Value Theorem.]*

If f has a periodic orbit then either it is a fixed point, in which case there is nothing to prove, or the smallest point x^- in the periodic orbit is distinct from the largest point x^+ in the periodic orbit. Now $f(x^-)$ lies in this periodic orbit, so $f(x^-) > x^-$, and $f(x^+)$ lies in this periodic orbit, so $f(x^+) < x^+$. Therefore the function g defined by $g(x) := f(x) - x$ is continuous, with $g(x^-) > 0$ and $g(x^+) < 0$, so by the Intermediate Value Theorem there exists $c \in (x^-, x^+)$ with $g(c) = 0$, therefore c is a fixed point of f.

Exercise 6. Given an iterated function system defined by the maps $\phi_1(x) = (x + 1)/10$ and $\phi_2(x) = (x + 4)/10$, define $\Phi(A) = \phi_1(A) \cup \phi_2(A)$, and let C_k denote $\Phi^k([0, 1])$ for $k \geq 0$.

(a) Determine the sets C_1 and C_2.

(b) If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k.

(c) What is the common length of each of the N_k closed intervals whose disjoint union equals C_k?

(d) Compute the box dimension of $C = \cap_{k=0}^{\infty} C_k$, being careful to justify your answer.

(e) Compute the box dimension of $D = \cap_{k=0}^{\infty} \Psi^k([0, 1])$, where $\Psi(A) = \psi_1(A) \cup \psi_2(A)$, and $\psi_1(x) = (x + 1)/16$, $\psi_2(x) = (x + 4)/16$.

(f) Describe a set E whose box dimension is equal to $4/5$, being careful to justify your answer.

(a)

$$C_1 = \left[\frac{1}{10}, \frac{2}{10}\right] \cup \left[\frac{4}{10}, \frac{5}{10}\right],$$

$$C_2 = \left[\frac{11}{100}, \frac{12}{100}\right] \cup \left[\frac{14}{100}, \frac{15}{100}\right] \cup \left[\frac{41}{100}, \frac{42}{100}\right] \cup \left[\frac{44}{100}, \frac{45}{100}\right].$$

(b) $N_k = 2^k$ because $N_0 = 1$ and the recursive procedure doubles the number of intervals at each step.

(c) The common length is 10^{-k}, because the length of the closed intervals decreases by a factor of 10 at each step, and the length of $C_0 = [0, 1]$ is 1.

(d) If $\varepsilon_k = 1/10^k$ then $N(\varepsilon_k) = 2^k$, so the box dimension equals

$$\lim_{k\to\infty} \frac{\log N(\varepsilon_k)}{-\log \varepsilon_k} = \lim_{k\to\infty} \frac{k \log 2}{k \log 10} = \frac{\log 2}{\log 10}.$$

(e) By analogy with the above calculation, at each step of the recursive procedure the number of intervals increases by a factor of $\beta = 2$, while the length of these intervals decreases by a factor of $\alpha = 1/16$, so the box dimension is equal to

$$\frac{\log \beta}{\log(1/\alpha)} = \frac{\log 2}{\log 16} = \frac{\log 2}{\log 2^4} = \frac{\log 2}{4 \log 2} = \frac{1}{4}.$$

(f) By analogy with the above calculation, it suffices to describe a recursive procedure where at each step the number of intervals increases by a factor of $\beta = 2^4$ and the length of these intervals decreases by a factor of $\alpha = 1/2^5$, since in that case the box dimension is equal to

$$\frac{\log \beta}{\log(1/\alpha)} = \frac{\log 2^4}{\log 2^5} = \frac{4 \log 2}{5 \log 2} = \frac{4}{5}.$$

Explicitly, we might define $\phi_j(x) = (x + 2j - 1)/2^5$ for $1 \le j \le 2^4$, then set

$$\Phi(A) = \bigcup_{j=1}^{2^4} \phi_j(A),$$

and define $E = \cap_{k=0}^{\infty} \Phi^k([0, 1])$.