MTH6107 Chaos & Fractals

Exercises 6

EXAM QUESTIONS: Exercises 1–5 below correspond to the various parts of Question 4 on the January 2023 exam paper, and Exercise 6 corresponds to Question 1 on the same exam paper.

Exercise 1. For the function $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \sum_{i=0}^{9} x^{2i+1}$, give a formula for the derivative $f_1'(x)$.

Exercise 2. Using properties of the derivative f_1', or otherwise, show that the only periodic point for f_1 is the fixed point at 0.

Exercise 3. For the function $f_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_2(x) = \begin{cases} -2(1+x) & \text{for } x < 0 \\ x - 2 & \text{for } x \geq 0\,, \end{cases}$$

evaluate the set $\{n \in \mathbb{N} : f_2 \text{ has a point of least period } n\}$, being careful to justify your answer.

Exercise 4. For the function $f_3 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_3(x) = \begin{cases} -2(1+x) & \text{for } x < 0 \\ x/2 - 2 & \text{for } x \geq 0\,, \end{cases}$$

evaluate the set $\{n \in \mathbb{N} : f_3 \text{ has a point of least period } n\}$, being careful to justify your answer.

Exercise 5. Without using Sharkovskii's Theorem, show that every continuous function $f : \mathbb{R} \to \mathbb{R}$ which has a periodic orbit must have a fixed point. [*Hint: Use the Intermediate Value Theorem.*]

Exercise 6. Given an iterated function system defined by the maps $\phi_1(x) = (x+1)/10$ and $\phi_2(x) = (x+4)/10$, define $\Phi(A) = \phi_1(A) \cup \phi_2(A)$, and let C_k denote $\Phi^k([0,1])$ for $k \geq 0$.

(a) Determine the sets C_1 and C_2.

(b) If C_k is expressed as a disjoint union of N_k closed intervals, compute the number N_k.

(c) What is the common length of each of the N_k closed intervals whose disjoint union equals C_k?

(d) Compute the box dimension of $C = \cap_{k=0}^{\infty} C_k$, being careful to justify your answer.

(e) Compute the box dimension of $D = \cap_{k=0}^{\infty} \Psi^k([0,1])$, where $\Psi(A) = \psi_1(A) \cup \psi_2(A)$, and $\psi_1(x) = (x+1)/16$, $\psi_2(x) = (x+4)/16$.

(f) Describe a set E whose box dimension is equal to $4/5$, being careful to justify your answer.