To justify the existence of such a dense orbit, we work symbolically, and let

$$s = 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111 \ldots$$

(where we build up s by first writing down all length-1 'words'/'blocks'

then all 〃 2 〃 〃

〃 3 〃 〃

〃 〃 4 〃 〃

⋮

〃 length-n 〃 〃

Define $x_0 = h(s)$

$$= \frac{0}{2} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{0}{2^4} + \frac{0}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \ldots$$

We claim x_0 has a dense orbit.

Now choose some (other) $x \in [0,1)$, and suppose x has binary expansion

$$b(x) = b_1 b_2 b_3 b_4 \ldots$$

(i.e. $h(b_1 b_2 b_3 \ldots) = x$).

Then it is possible (due to the recipe for defining s) to choose some $i \in \mathbb{N}$ such that $\sigma^i(s)$ and $b(x)$ begin with a common (long) block of digits/symbols

i.e. $\sigma^i(s) = \underbrace{b_1 b_2 \ldots b_k}_{\text{same symbols as } b(x) \text{ begins with}} c_{k+1} c_{k+2} \ldots$

This ensures that

$$|D^i(x_0) - x| < \frac{1}{2^k} < \varepsilon \quad \text{if}$$

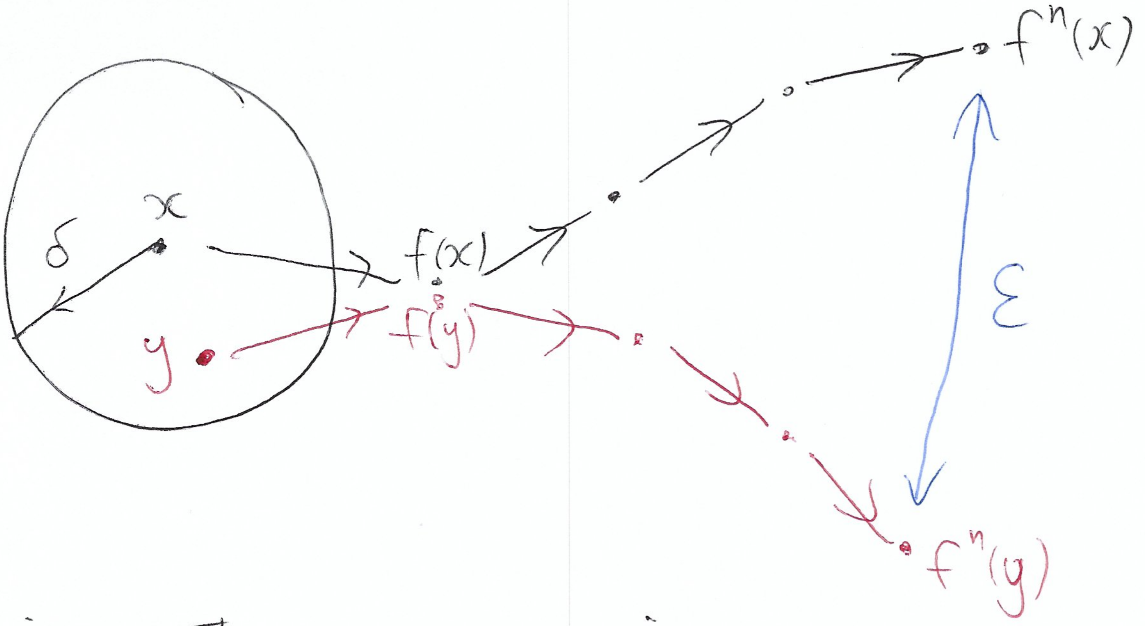
k is chosen sufficiently large.

So x_0 <u>does</u> have a dense orbit under D.

The tent map T, and the logistic map f_4 both also have the property that there are points with dense orbits.

Although there is no agreed definition of 'chaos', there is some universal agreement that an indication of chaotic dynamics is sensitive dependence on initial conditions (SDIC):

Defn Let $I \subset \mathbb{R}$ be an interval, and consider $f: I \to I$. We say that f has sensitive dependence on initial conditions (SDIC) at $x \in I$ if there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $y \in I$ and $n \in \mathbb{N}$ such that $|x-y| < \delta$ and $|f^n(x) - f^n(y)| > \varepsilon$.



i.e. There are points arbitrarily close to x whose orbits separate/diverge from the orbit of x (by at least ε)

<u>Proposition</u> The doubling map $D: [0,1) \circlearrowleft$ (given by $D(x) = 2x \pmod 1$) has SDIC at <u>all</u> $x \in [0,1)$

<u>Proof</u> Let $\varepsilon = 1/3$. Suppose $x \in [0,1)$. Given $\delta > 0$, choose $n \in \mathbb{N}$ such that $\frac{1}{2^{n+1}} < \delta$ and let

$$y = \begin{cases} x + \frac{1}{2^{n+1}} & \text{if } D^n(x) \in [0, \tfrac{1}{2}) \\ x - \frac{1}{2^{n+1}} & \text{if } D^n(x) \in [\tfrac{1}{2}, 1) \end{cases}$$

Then $|y-x| = \frac{1}{2^{n+1}} < \delta$, and

$|D^n(y) - D^n(x)| = \frac{1}{2} > \frac{1}{3} = \varepsilon$. $\square$

<u>Defn</u> If f has SDIC at every $x \in I$ then we say that f has <u>sensitive dependence on initial conditions</u> (SDIC).

It can also be shown that:

<u>Prop</u> The tent map $T : [0,1] \to [0,1]$ (given by $T(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2-2x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$) has SDIC.

<u>Lemma</u> If $f: I \to I$ has SDIC, and is topologically conjugate to some $g: I \to I$, then g has SDIC

<u>Consequently</u>:

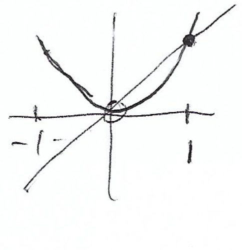
<u>Prop</u> The logistic map $f_4 : [0,1] \to [0,1]$ (given by $f_4(x) = 4x(1-x)$) has SDIC.

<u>Remark</u>

(a) A map of the form $f(x) = x + c$ $(c \in \mathbb{R})$ does <u>not</u> have SDIC at <u>any</u> point

(if $|x-y| = \delta$ then $|f^n(x) - f^n(y)| = \delta$ for all $n \geq 0$)

(b) The map $f(x) = x^2$ does <u>not</u> have SDIC at any point $x \in (-1,1)$



Definition (from Robert Devaney's book "An Introduction to Chaotic Dynamical Systems")

Let $I \subset \mathbb{R}$. We say that $f : I \circlearrowleft$ is chaotic (in the sense of Devaney) if

(i) f has SDIC (at all points in I)

(ii) the set of periodic points of f is a dense subset of I

(iii) there is an orbit (under f) of some point which is a dense subset of I

This definition of Devaney dates back to the mid-1980s

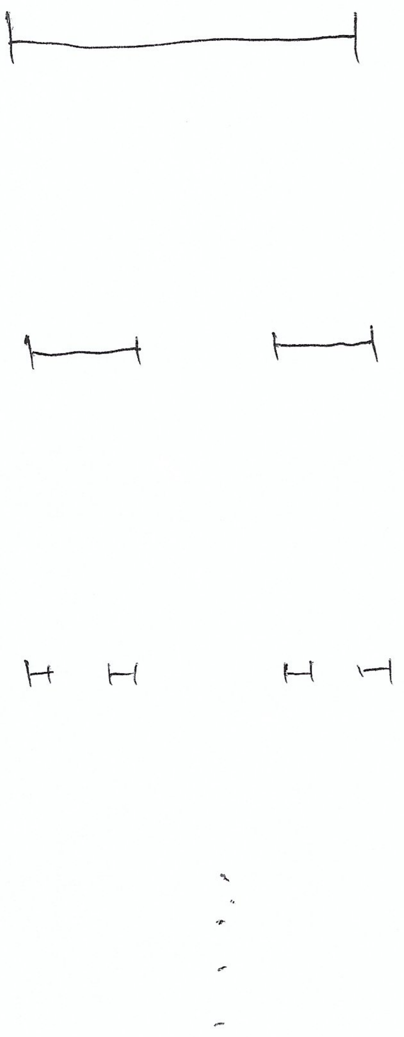
It was later shown (in the mid-1990s) that if $f : I \to I$ is continuous then property (iii) implies (i) and (ii).

So the definition of 'chaotic' (in the sense of Devaney) can be reduced to saying that some point has a dense orbit under f (<u>if</u> f is continuous).

Examples

- The doubling map $D : [0,1) \to [0,1)$ is chaotic in the sense of Devaney.
- The tent map $T : [0,1] \to [0,1]$ is chaotic in the sense of Devaney
- The logistic map $f_4 : [0,1] \to [0,1]$ is chaotic in the sense of Devaney.

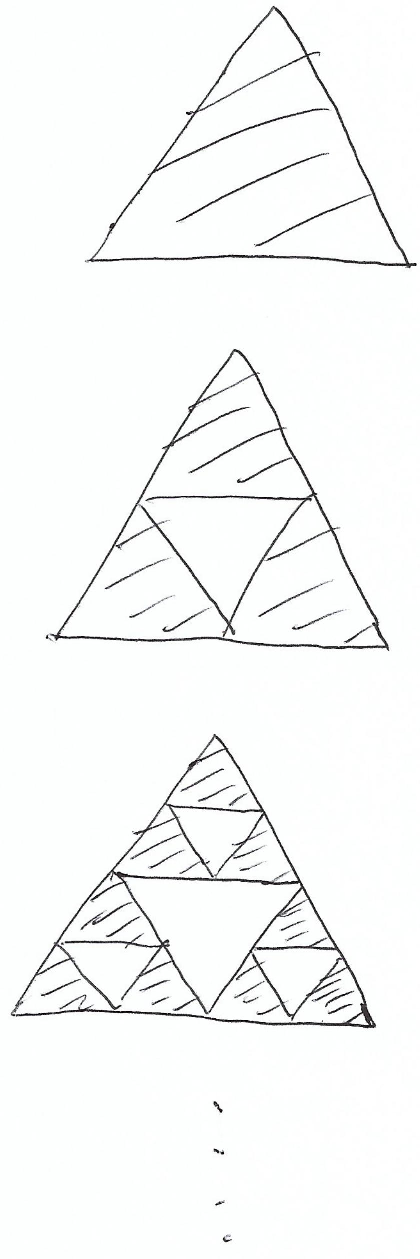
Fractals   Pictorial Motivation



Step 0

Step 1

Step 2



"Cantor Set"

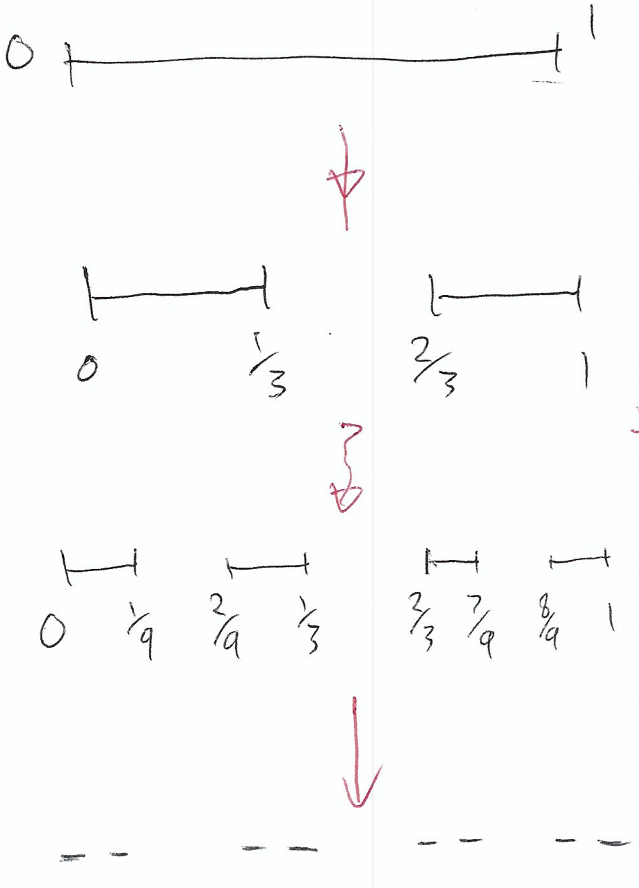
Sierpinski Triangle

These objects are obtained "in the limit" as the number of steps tends to ∞

Cantor set

We shall now construct the *middle-third Cantor set*.

Let $C_0 = [0,1] \subset \mathbb{R}$, and consider the following inductive definition of C_n:



$C_0 = [0,1]$

(Remove the open 'middle third' $(1/3, 2/3)$ of this interval)

$C_1 = [0, 1/3] \cup [2/3, 1]$

Remove the open middle third of the two disjoint intervals making up C_1

$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$

C_3 = union of 8 disjoint closed intervals

We continue this indefinitely, and at step n:

$$C_n = C_{n-1} \setminus \left\{ \begin{array}{l} \text{open middle thirds} \\ \text{of all sub-intervals of} \\ C_{n-1} \end{array} \right\}$$

Note that C_n is a union of 2^n disjoint closed intervals of the form

$$\left[\frac{k}{3^n}, \frac{k+1}{3^n}\right] \quad \left(\text{these intervals have length } 1/3^n\right)$$

<u>Definition</u> The <u>middle third Cantor set</u> is $C = \bigcap_{n=0}^{\infty} C_n$

(= the set of points which lie in all of the C_n, i.e. never lie in the open middle third of any sub-interval)

Observe that the total 'length' of C_n is

$$\left(\tfrac{2}{3}\right)^n, \quad \text{which} \to 0 \text{ as}$$

$n \to \infty$, so C has length 0.

A more general notion:

<u>Defn</u> A set of the form $\varphi(C)$, where C is the middle third Cantor set, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, is called a <u>Cantor set</u>.

Cantor sets K have the following properties:
(which actually could be regarded as <u>defining</u> properties of the notion of Cantor set)

1. K is 'perfect', i.e. each point $x \in K$ has other points in K which are arbitrarily close to it.

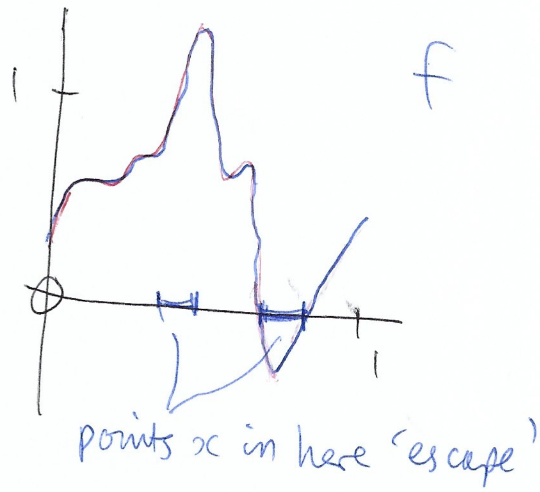
2. K is 'totally disconnected', i.e. K contains no non-empty open intervals

3. K is 'closed' (i.e. every sequence of points in K which converges to a limit l, is such that $l \in K$)

<u>Qu</u> How is this discussion ~~of~~ related to dynamical systems?

Defn

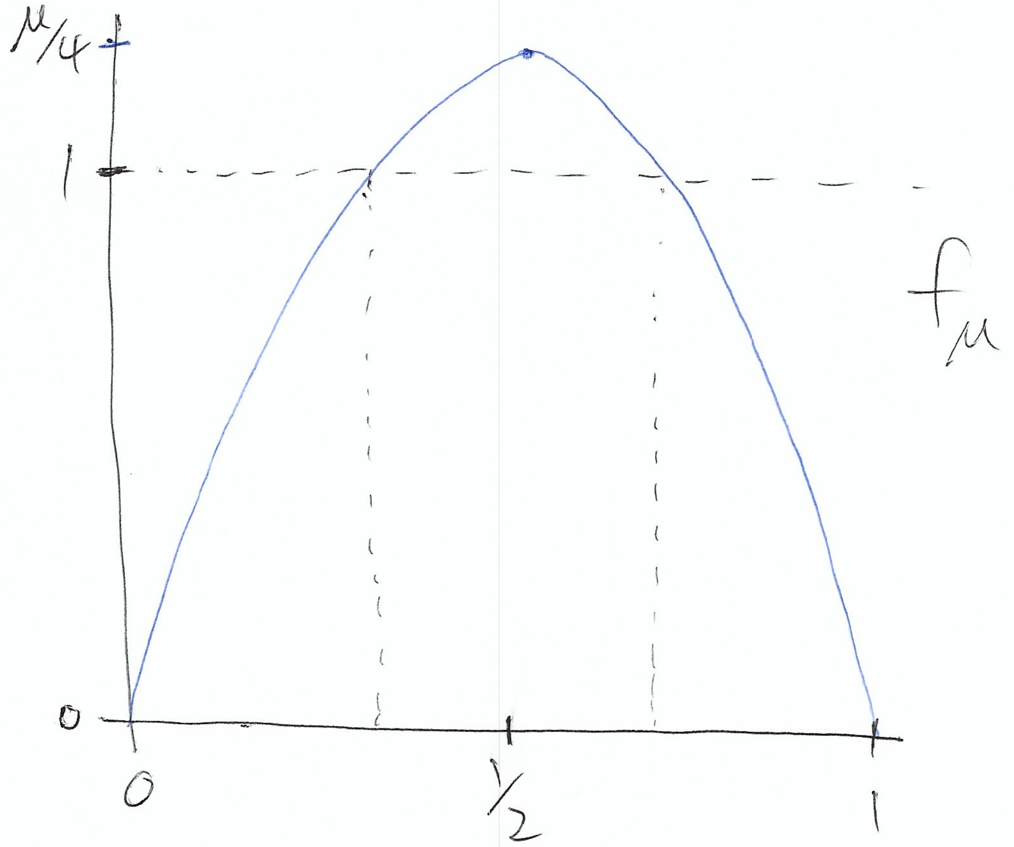
For a map $f:[0,1] \to \mathbb{R}$, a point $x \in [0,1]$ is said to be **non-escaping** if $f^n(x) \in [0,1]$ for all $n \geq 0$.



The set $\Lambda = \Lambda(f) := \{\text{non-escaping points}\}$ is called the **set of non-escaping points**, or simply the **non-escaping set**.

The non-escaping set of the logistic map $f_\mu : [0,1] \to \mathbb{R}$, defined by $f_\mu(x) = \mu x(1-x)$, turns out to be a Cantor set for $\mu > 4$.

[Recall, previously we only considered the parameter μ satisfying $0 \le \mu \le 4$]



Notice that since $\mu > 4$, the graph of f_μ rises above the line $y = 1$ in a central interval around $x = \frac{1}{2}$.

All points in this interval "escape" from $I = [0,1]$ after a single iterate, and therefore do not belong to $\Lambda(f_\mu)$.