Week 9

What we'll cover (lecture 1)

A] Eigenvalues: characteristic polynomial

B] Some important properties of the characteristic polynomial and the eigenvectors

A] The following statements are equivalent

(a) λ is an eigenvalue of a $n \times n$ matrix A

(b) $(A - \lambda I)\underline{x} = 0$ has a non-trivial solution

(c) The nullspace $N(A - \lambda I)$ is non-trivial

(d) $(A - \lambda I)$ is singular

(e) $\det(A - \lambda I) = 0$

(a) $\Rightarrow$ (b) $\quad A\underline{x} = \lambda \underline{x} \Rightarrow 0 = A\underline{x} - \lambda \underline{x} = A\underline{x} - \lambda I \underline{x} \Rightarrow$

with $\underline{x} \neq 0 \qquad (A - \lambda I)\underline{x} = 0$

(b) $\Rightarrow$ (c) obvious, (d) and (e) follow from the Invertible Matrix theorem

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It is useful to start from statement (e) and look at the determinant of $(A - \lambda I)$ as a function of λ. By construction it is a polynomial $P_A(\lambda)$ of degree n called characteristic polynomial of A: $P_A(\lambda) = \det(A - \lambda I)$ (definition)

The characteristic equation is $P_A(\lambda) = 0$ and

Root of the characteristic equation are exactly the eigenvalues of A

Remarks:

(i) A $n \times n$ matrix can have at most n different eigenvalues (since $P_A(\lambda)$ has degree n)

(ii) A $n \times n$ real matrix may not have n eigenvalues

Examples: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has no real eigenvalues; $\lambda = 1$ is the only eigenvalue for the identity matrix I.

$$P_A(\lambda) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 \Rightarrow P_A(\lambda) = 0 \Leftrightarrow \lambda^2 = -1$$

(Instead there always is one complex eigenvalue, thanks to the "Fundamental Theorem of Algebra").

- Once you have found an eigenvalue λ, you can find a corresponding eigenvector by solving the linear system $A\underline{x} = \lambda \underline{x}$ for the particular λ you are considering.

Ex 1] $A = \begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix}$

$$P_\lambda(A) = \det \begin{pmatrix} -7-\lambda & -6 \\ 9 & 8-\lambda \end{pmatrix} = (-7-\lambda)(8-\lambda) + 54;$$

$$P_\lambda(A) = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda-2)(\lambda+1) = 0$$

Clearly there are two eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$

Let us look for the corresponding eigenvectors

$\lambda_1 = -1$. $(A - \lambda_1 I)\underline{x} = 0$: the matrix associated to this linear system is

$$\left(\begin{array}{cc|c} -7+1 & -6 & 0 \\ 9 & 8+1 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -6 & -6 & 0 \\ 9 & 9 & 0 \end{array}\right) \overset{REF}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Thus a solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Check $\begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ✓

The null space of $(A + \lambda_1 I)$ is $\text{span}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) = \left\{\begin{pmatrix} -\alpha \\ \alpha \end{pmatrix}\right\}$

This is the eigenspace corresponding to λ_1 and any non-trivial vector in this space is an eigenvector corresponding to $\lambda_1 = 1$.

Similarly for $\lambda_2 = 2$ we have

$$\left(\begin{array}{cc|c} -7-2 & -6 & 0 \\ 9 & 8-2 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -9 & -6 & 0 \\ 9 & 6 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Thus a solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and

$\text{Span}\left(\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right) = \left\{\begin{pmatrix} 2\alpha \\ -3\alpha \end{pmatrix}\right\}$ is the corresponding eigenspace.

check: $\begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -14+18 \\ 18-24 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix} = 2\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ ✓

<u>Comment 1</u>: Different matrices can have the same charcteristic polynomial. Consider for instance

$$A' = \begin{pmatrix} -7 & +6 \\ -9 & 8 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

We have $\det(A' - \lambda I) = \det(A - \lambda I)$

B] This is a general property of similar matrices

Recall A' and A are similar iff there exists a matrix P such that $A' = P^{-1} A P$ (in the example above $P = P^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

<u>Theorem</u>: Similar matrices have the same characteristic polynomial. Let $A' = P^{-1} A P$, we have

$$P_{A'}(\lambda) = \det(A' - \lambda I) = \det(P^{-1} A P - \lambda I) =$$
$$\det[P^{-1}(A - \lambda I) P] = \det(P^{-1}) \underbrace{\det(A - \lambda I)}_{P_A(\lambda)} \det(P)$$

Since $\det(P^{-1}) \det(P) = \det(P^{-1}P) = \det I = 1$

we have $P_{A'}(\lambda) = P_A(\lambda)$ ✓

<u>Comment 2</u>: notice that the eigenvectors of the λ_1 eigenspace are linearly independent from those of the λ_2 eigenspace. Example

$$\underline{x}_{\lambda_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} ; \ \underline{x}_{\lambda_2} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \Rightarrow \alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 = 0$$

iff $\alpha_i = 0$ (for instance $\det \begin{pmatrix} -1 & 2 \\ 1 & -3 \end{pmatrix} = 1 \neq 0$).

This is a general property

Theorem: If $\underline{v}_1, \dots, \underline{v}_n$ are eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with $\lambda_i \neq \lambda_j$ $\forall i, j$, then $\{\underline{v}_1, \dots, \underline{v}_n\}$ are linearly independent.

Proof by induction. Start from $n = 2$ and consider $\underline{v} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2$. We have either $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$ (or both)

$$A(\underline{v}) = \alpha_1 A(\underline{v}_1) + \alpha_2 A(\underline{v}_2) = \alpha_1 \lambda_1 \underline{v}_1 + \alpha_2 \lambda_2 \underline{v}_2$$

Should $\underline{v} = 0$ for some α_i, we'd have

$$\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 = 0$$

$$\alpha_1 \lambda_1 \underline{v}_1 + \alpha_2 \lambda_2 \underline{v}_2 = A(\underline{v} = 0) = 0$$

multiply the 1st eq. by the $\lambda_i \neq 0$ and subtract the 2nd eq. $\Rightarrow$

$$\alpha_1 (\lambda_2 - \lambda_1) \underline{v}_1 = 0$$

or

$$\alpha_2 (\lambda_1 - \lambda_2) \underline{v}_2 = 0$$

Since by hypothesis $\lambda_1 \neq \lambda_2$ and $\underline{v}_1 \neq 0$, $\underline{v}_2 \neq 0$ then both equation on the right holds: for instance, the first one implies $\alpha_1 = 0$ and from $\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 = 0$ we get $\alpha_2 = 0$ which is the content of the second one. Thus $\underline{v} = 0$ iff so the $\underline{v}_i$'s are linearly independent.

Then suppose that the theorem holds for $\underline{v}_1, \dots, \underline{v}_{n-1}$. You can prove that it holds for $\underline{v}_1, \dots, \underline{v}_n$ by using a similar approach as above

$\underline{v} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n$. If $\underline{v} = 0$ then

$$\begin{aligned} \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n &= 0 \\ \alpha_1 \lambda_1 \underline{v}_1 + \dots + \alpha_n \lambda_n \underline{v}_n &= 0 \end{aligned} \Rightarrow \sum_{i=1}^{n-1} \alpha_i (\lambda_n - \lambda_i) \underline{v}_i = 0$$

Since $\lambda_n - \lambda_i \neq 0$ and $\underline{v}_i \neq 0$ $\forall i$ then $\alpha_1 = \dots = \alpha_{n-1} = 0$

which implies $\alpha_n = 0$ and so $\{\underline{v}_1, \dots, \underline{v}_n\}$ are linearly independent.

Comment: coming back to the example above, we have

$\underline{x}_{\lambda_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\underline{x}_{\lambda_2} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$. Consider $P = \begin{pmatrix} -1 & 2 \\ 1 & -3 \end{pmatrix}$, i.e.

a matrix whose column vectors are linearly independent eigenvectors. In our case $\text{rank}(P) = \dim(V) = 2$.

P is invertible $P^{-1} = -\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$. We have

$$P^{-1}AP = -\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -7 & -6 \\ 9 & 8 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -3 \end{pmatrix} =$$

$$-\begin{pmatrix} -3 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix},$$ i.e $P^{-1}AP$ is

a diagonal matrix whose entries are the eigenvectors!

What we'll cover (lecture 2→3)

A] Diagonalization

B] Eigenvalues / eigenvectors of abstract linear maps

C] New structures for vector spaces

Let us see why we should expect that the matrix $P^{-1}AP$ above is diagonal. We have

$$(e_1)\quad P\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1 & 2\\ 1 & -3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} \leftarrow \text{an eigenvector for } \lambda_1$$

$$(e_2)\quad P\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1 & 2\\ 1 & -3\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\-3\end{pmatrix} \leftarrow \text{an eigenvector for } \lambda_2$$

In other worlds P is the transition matrix from the eigenvector basis $E = \{\underline{v}_1, \underline{v}_2\}$ to the canonical basis $C = \{\underline{x}_1, \underline{x}_2\}$ of $\mathbb{R}^2$: $P = [id]^E_C = P_{E,C}$

Then P takes the coordinates of $\underline{v}_1$ according to the basis E (i.e. $\begin{pmatrix}1\\0\end{pmatrix}$) to the coordinates of the same vector according to the basis C (where A reads $\begin{pmatrix}-7 & -6\\ 8 & 9\end{pmatrix}$), see (e_1). Similarly for $\underline{v}_2$.

So $A' = P^{-1}AP$ is just an example of the change of basis formula see in Week 8

$$P^{-1} A P \longleftrightarrow [id]_E^C [A]_C^C [id]_C^E = [A]_E^E \rightarrow A'_{ij}$$

seeing A' as the same linear transformation defined by A, but now in the E-basis. In this basis we have

$$A \underline{v}_1 = \lambda_1 \underline{v}_1 \xrightarrow{\text{E-basis}} A' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} \lambda_1 & * \\ 0 & * \end{pmatrix}$$

$$A \underline{v}_2 = \lambda_2 \underline{v}_2 \xrightarrow{\text{E-basis}} A' \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} * & 0 \\ * & \lambda_2 \end{pmatrix}$$

Thus $A' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\Rightarrow$ This is a general property, which I state without a formal proof. <u>Theorem</u>: a $n \times n$ matrix is diagonalisable (i.e. it is similar to a diagonal matrix) iff it has n-linearly independent vectors:

$$P^{-1} A P = D \leftarrow \text{diagonal}$$

The column vectors of P are the eigenvectors of A (in the C-basis) and the non-trivial entries of D are the eigenvalues.

Ex 1]

$$A = \begin{pmatrix} -7 & 3 & -3 \\ -9 & 5 & -3 \\ 9 & -3 & 5 \end{pmatrix}$$

$$P_A(\lambda) = \det \begin{pmatrix} -7-\lambda & 3 & -3 \\ -9 & 5-\lambda & -3 \\ 9 & -3 & 5-\lambda \end{pmatrix} =$$

equal

$$(-7-\lambda)\left((5-\lambda)^2 - 9\right) - 3\left(-9(5-\lambda) + 27\right) - 3\left(+27 - 9(5-\lambda)\right) =$$

$$(-7-\lambda)\left((5-\lambda)+3\right)\left((5-\lambda)-3\right) + 54\left((5-\lambda)-3\right) =$$

$$(2-\lambda)\left((-7-\lambda)(8-\lambda)+54\right) = (2-\lambda)(\lambda^2-\lambda-2) =$$

$$(2-\lambda)\left((\lambda-2)(\lambda+1)\right) = -(\lambda-2)^2(\lambda+1)$$

The eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 2$

The eigenevectors corresponding to λ_1 satisfy

$(A+I)\underline{v} = 0 \Rightarrow$ non-trivial elements of $N(A+I)$

$$\left(\begin{array}{ccc|c} -7-\lambda_1 & 3 & -3 & 0 \\ -9 & 5-\lambda_1 & -3 & 0 \\ 9 & -3 & 5-\lambda_1 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -6 & 3 & -3 & 0 \\ -9 & 6 & -3 & 0 \\ 9 & -3 & 6 & 0 \end{array}\right) \overset{REF}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\underline{x} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ is a solution and $N(A+I) = \text{span}\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\}$

Similarly for λ_2 we have $(A-2I)\underline{v} = 0 \Rightarrow$

$$\begin{pmatrix} -7-\lambda_2 & 3 & -3 & | & 0 \\ -9 & 5-\lambda_2 & -3 & | & 0 \\ 9 & -3 & 5-\lambda_2 & | & 0 \end{pmatrix} = \begin{pmatrix} -9 & 3 & -3 & | & 0 \\ -9 & 3 & -3 & | & 0 \\ 9 & -3 & 3 & | & 0 \end{pmatrix} \overset{REF}{\sim} \begin{pmatrix} 1 & -1/3 & 1/3 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Then $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ are independent solutions

and $N(A-2I) = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \right\}$

We have 3 linearly independent eigenvectors so

A is diagonalisable

$$A' = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 2/3 & -4/3 \\ 1 & -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} -7 & 3 & -3 \\ -9 & 5 & -3 \\ 9 & -3 & 5 \end{pmatrix} \begin{pmatrix} -1 & 1 & -1 \\ -1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Ex 2 Are all matrices diagonalisable? NO!

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\qquad P_A(\lambda) = (1-\lambda)^2$

It has one eigenvalue $\lambda_1 = 1$. The eigenevectors are in $N(A-I) = N\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This space is one-dimensional $N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

So there is just one eigenvector

Ex 3:

$$A = \begin{pmatrix} -6 & 3 & -2 \\ -7 & 5 & -1 \\ 8 & -3 & 4 \end{pmatrix} \Rightarrow P_A(\lambda) = \det\begin{pmatrix} -6-\lambda & 3 & -2 \\ -7 & 5-\lambda & -1 \\ 8 & -3 & 4-\lambda \end{pmatrix}$$

$$P_A(\lambda) = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda+1)(\lambda^2 - 4\lambda + 4) =$$

$\lambda = -1$ is a solution

$$= -(\lambda+1)(\lambda-2)^2 \ \ldots \text{ same } P_A(\lambda) \text{ as before!}$$

However $N(A-2I)$ is obtained from

$$\left(\begin{array}{ccc|c} -8 & 3 & -2 & 0 \\ -7 & 3 & -1 & 0 \\ 8 & -3 & 2 & 0 \end{array}\right) \overset{REF}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Only one independent solution $N(A-2I) = \text{Span}\left\{\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}\right\}$

Thus A it is not diagonalisable!

It is still interesting to write A in a basis that contains as many eigenvectors as possible

$N(A-\lambda_1 I)$ follows from $\left(\begin{array}{ccc|c} -5 & 3 & -2 & 0 \\ -7 & 6 & 1 & 0 \\ 8 & -3 & 5 & 0 \end{array}\right) \overset{REF}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$N(A+I) = \text{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$ ← not an eigenvector

Choose $E = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

$$A' = P^{-1} A P = \begin{pmatrix} -1 & 0 & 12 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{with } P = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

The matrix A' is upper triangular!
(We'll come back to this point...)

Theorem: The eigenvalues of a triangular matrix are its diagonal entries

The proof is straightforward from $P_A(\lambda)$

B] Let us consider an abstract real vector space V and a linear map $L: V \to V$. We defined the eigenvector/eigenvalue equation in this case as well

$L\underline{v} = \lambda \underline{v}$ with $\underline{v} \in V$ & $\underline{v} \neq 0$, $\lambda \in \mathbb{R}$

We can find the eigenvalues simply introducing a basis C for V and solving the problem for the associated matrix $A = [L]_C^C$.

Remark: The eigenvalues of A do not depend on the choice of basis; different choices yield similar matrices and so the same charcteristic polynomial. Then one can find the eigenvectors $\underline{x}_i$ of A as above and reconstruct the abstract vectors $\underline{v}_i$ such that $[\underline{v}_i]_C = \underline{x}_i$

c] Let us now introduce a new structure defined on the elements of a real vector space V: the norm.

Definition: A norm is map from V to $\mathbb{R}$ satisfying the following properties

(a1) • the norm of $\underline{v} \in V$ (indicated by $\|v\|$) is non-negative $\|v\| \geq 0 \quad \forall \underline{v} \in V$

(a2) • $\|\underline{v}\| = 0$ iff $\underline{v} = \underline{0}$

(a3) • $\|\alpha \underline{v}\| = |\alpha| \, \|\underline{v}\|$ for any $\underline{v} \in V$ and $\alpha \in \mathbb{R}$

(a4) • $\|\underline{v} + \underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\|$ for any $\underline{v}, \underline{w} \in V$ (triangle inequality)

Ex 1] A norm in $\mathbb{R}^2$, $\underline{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, is

$$\|\underline{x}\| = \left(x_1^2 + x_2^2\right)^{1/2}$$

(a1)–(a3) obvious. (a4) see later

Ex 2] Another norm in $\mathbb{R}^2$ $\|x\| = |x_1| + |x_2|$

(a1)–(a3) obvious. (a4) let $\underset{\sim}{x}, \underline{y} \in \mathbb{R}^2$, we have

$$\|\underline{x} - \underline{y}\| = \left\| \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \end{pmatrix} \right\| = |x_1 - y_1| + |x_2 - y_2| \leq$$

$$\left(|x_1| + |y_1|\right) + \left(|x_2| + |y_2|\right) \leq \|\underline{x}\| + \|\underline{y}\| \quad \checkmark$$

Definition: a unit vector is a vector with norm 1

If $\underline{v} \neq \underline{0}$ then $\frac{1}{\|v\|} \underline{v}$ is a unit vector (check it!)

Definition: the distance between $\underline{x}, \underline{y} \in \mathbb{R}^n$ with respect to the norm $\|.\|$ is $d(\underline{x}, \underline{y}) = \|\underline{x} - \underline{y}\|$.

Sometime I'll refer to $\|\underline{x} - \underline{y}\|$ also as the length of $\underline{x} - \underline{y}$

A finer structure is that of an inner product of a (real) vector space V. It is a map from

$V \times V$ to $\mathbb{R}$ (indicated with $\langle \underline{v}, \underline{w} \rangle \in \mathbb{R}$ and $\underline{v}, \underline{w} \in V$) with the following properties

(b1) • $\langle \underline{v}, \underline{w} \rangle = \langle \underline{w}, \underline{v} \rangle$

(b2) • $\langle \underline{v}_1 + \underline{v}_2, \underline{w} \rangle = \langle \underline{v}_1, \underline{w} \rangle + \langle \underline{v}_2, \underline{w} \rangle$

(b3) • $\langle \alpha \underline{v}, \underline{w} \rangle = \alpha \langle \underline{v}, \underline{w} \rangle$

(b4) • $\langle \underline{v}, \underline{v} \rangle \geq 0 \quad \forall \underline{v} \in V$

(b5) • $\langle \underline{v}, \underline{v} \rangle = 0$ iff $\underline{v} = \underline{0}$