Machine Learning with Python

MTH786U/P 2023/24

Week 11: Semi-supervised classification with graphs

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What is a graph?

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An *undirected graph* is a pair $G = (V, E)$

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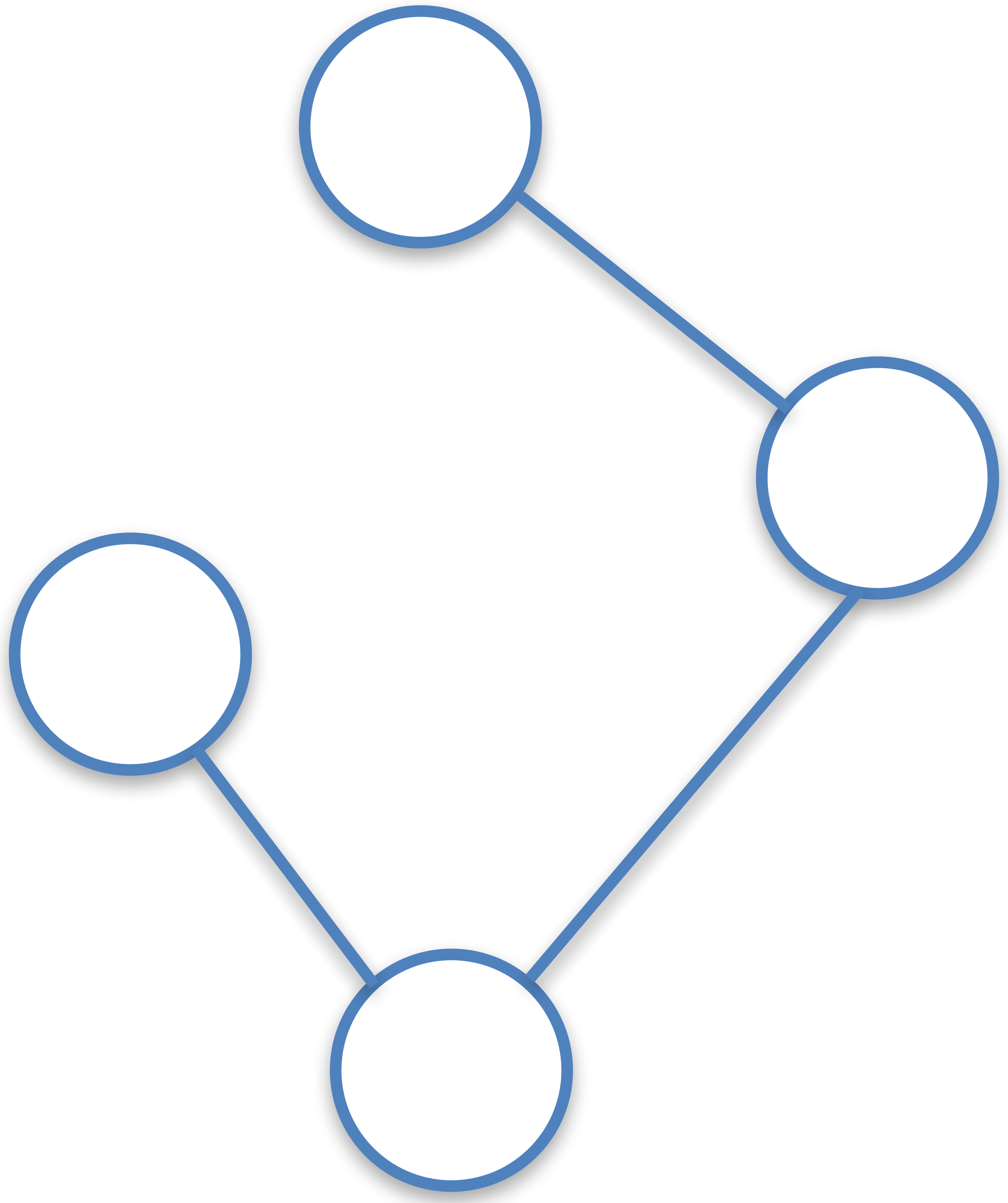

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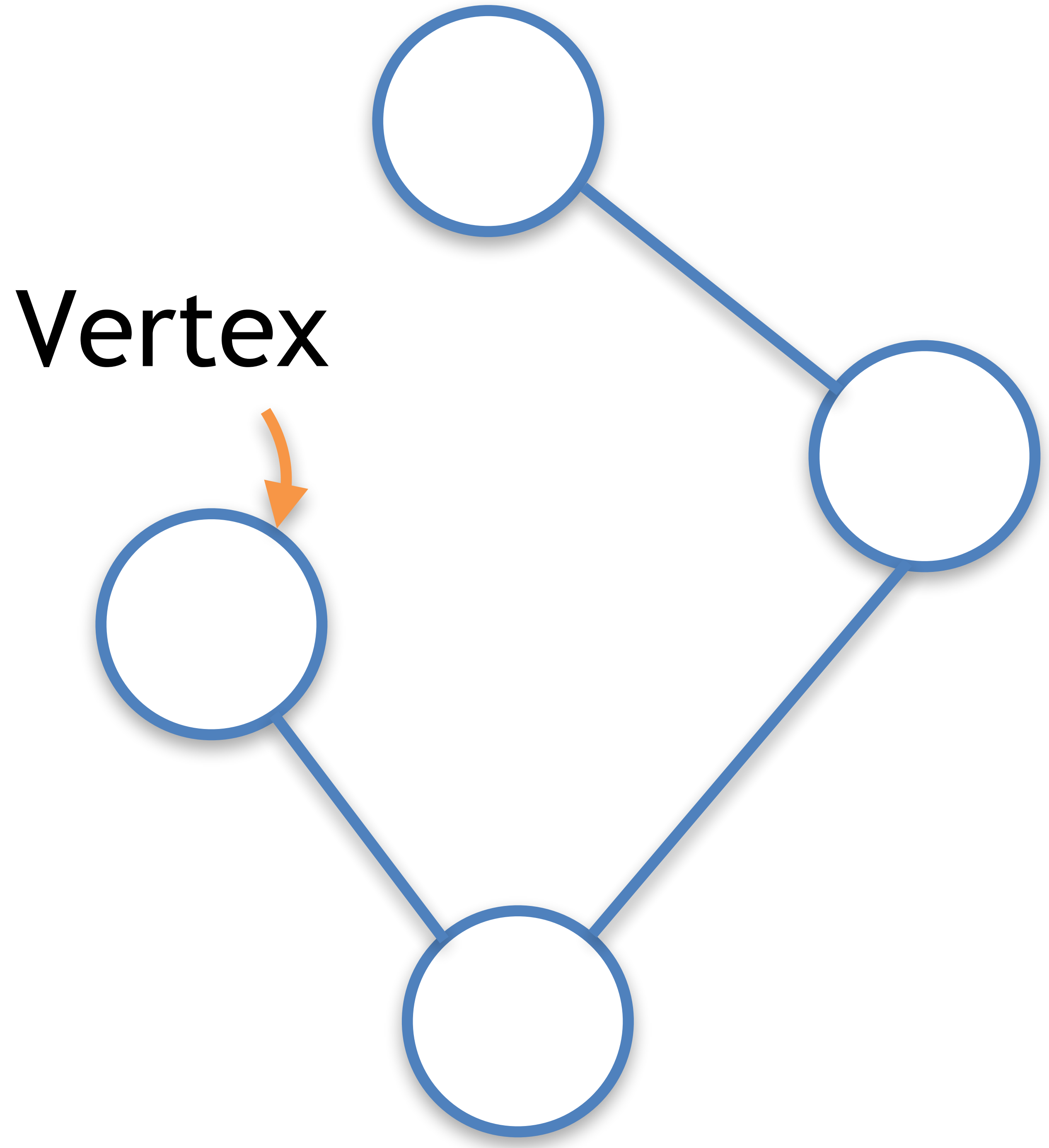

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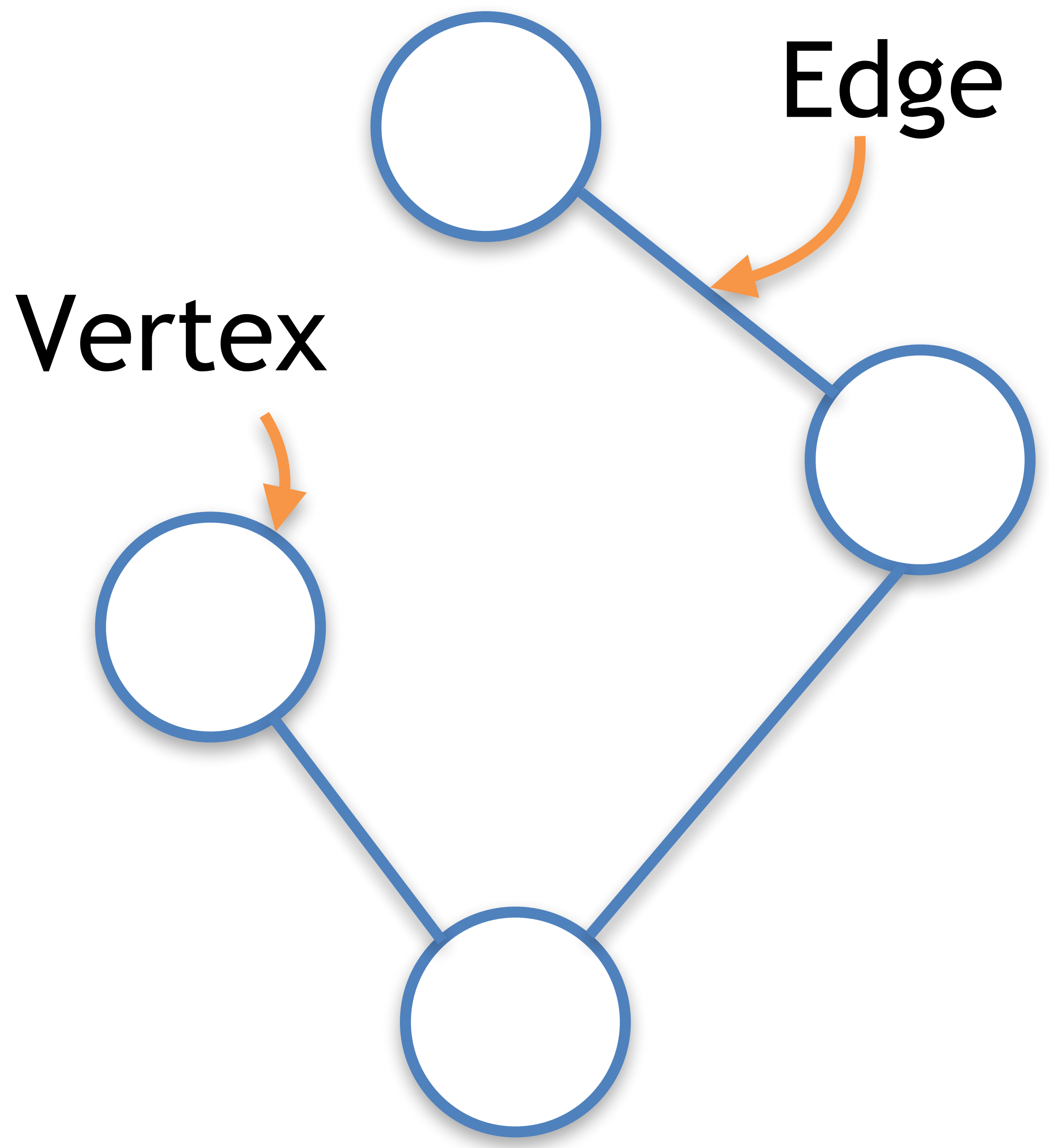

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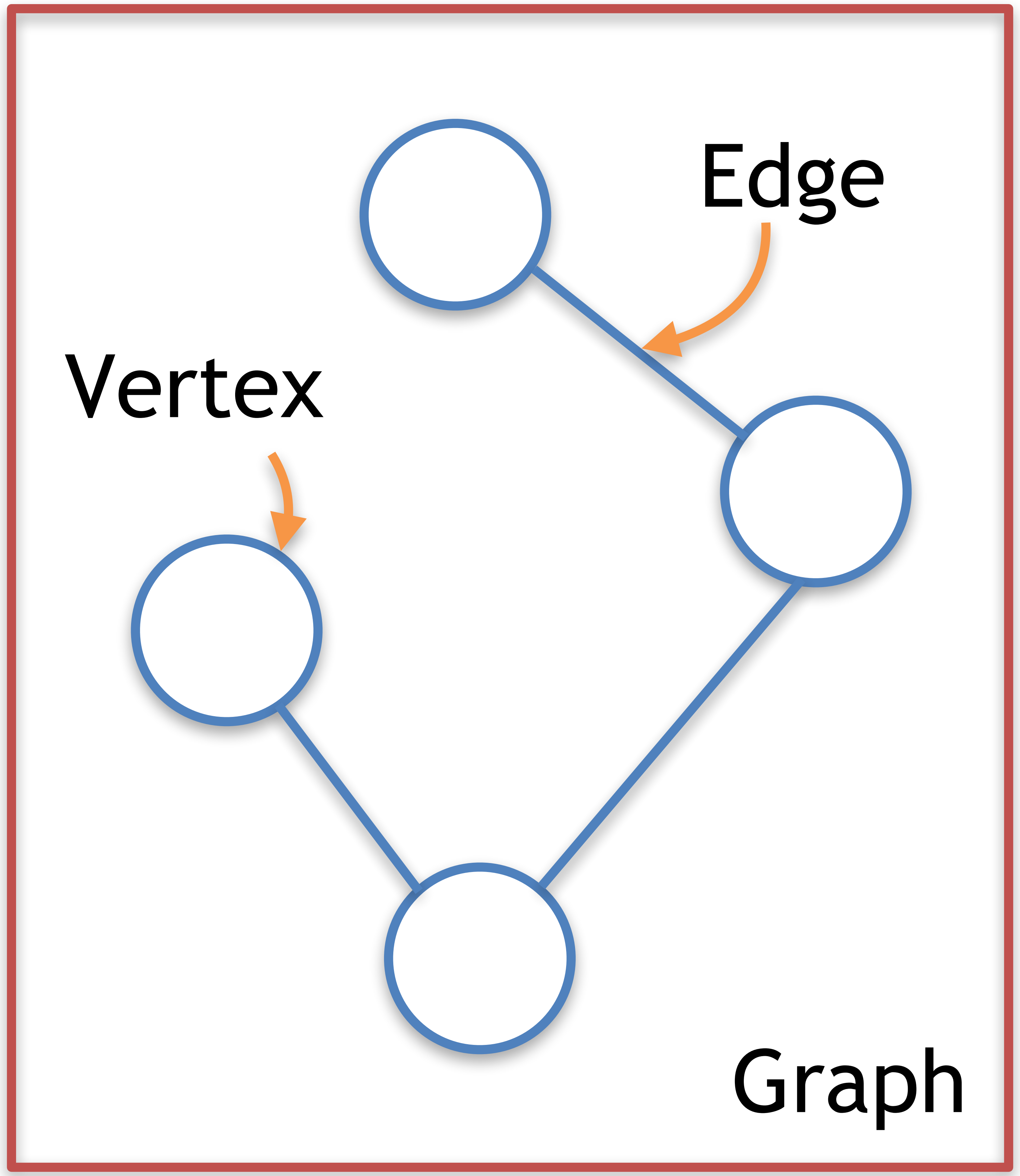

What is a graph?

A *weighted graph* is a pair $G = (V, E)$

V are the *vertices*

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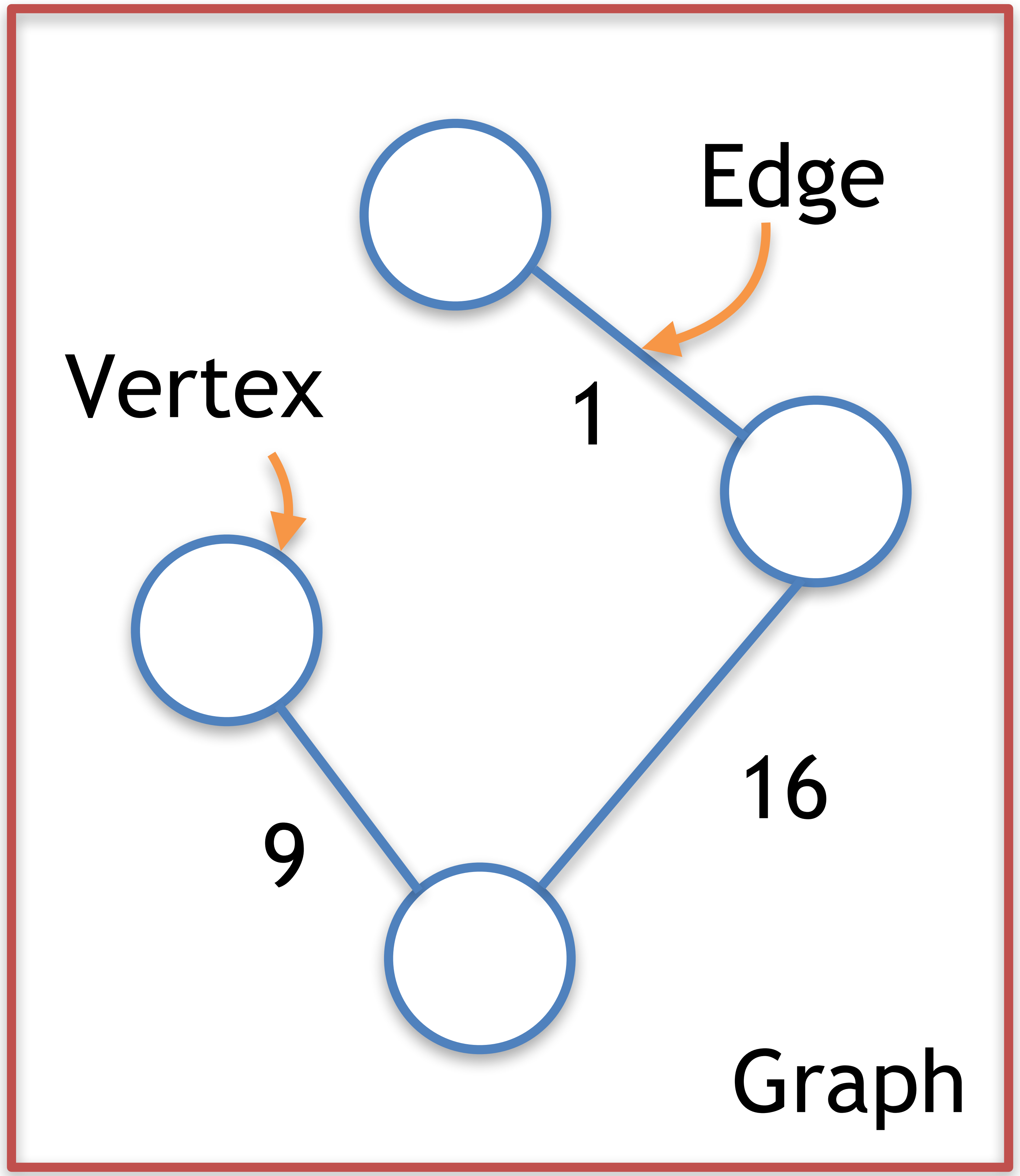
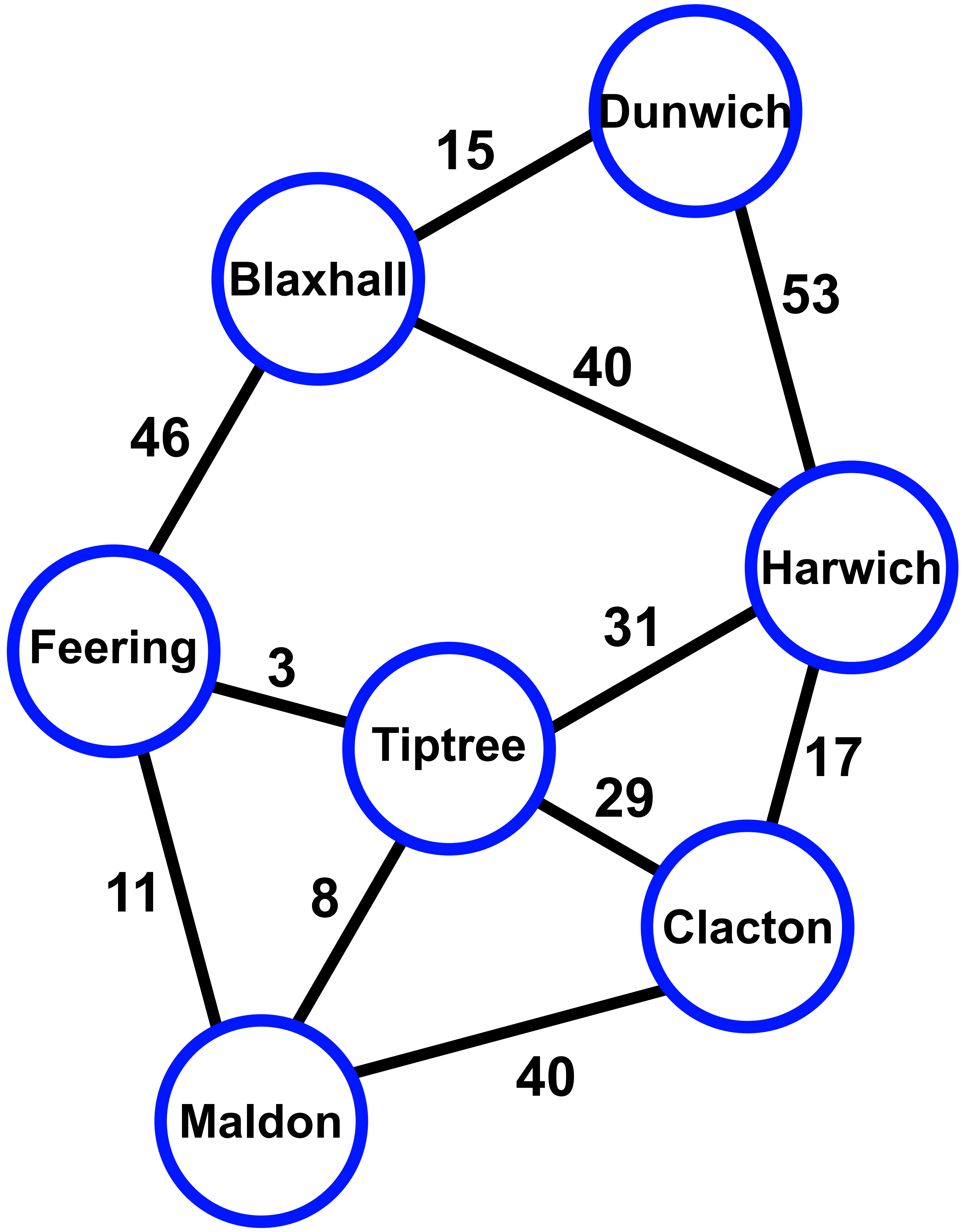
A number called *weight* is assigned to each edge



What is a graph?

Example



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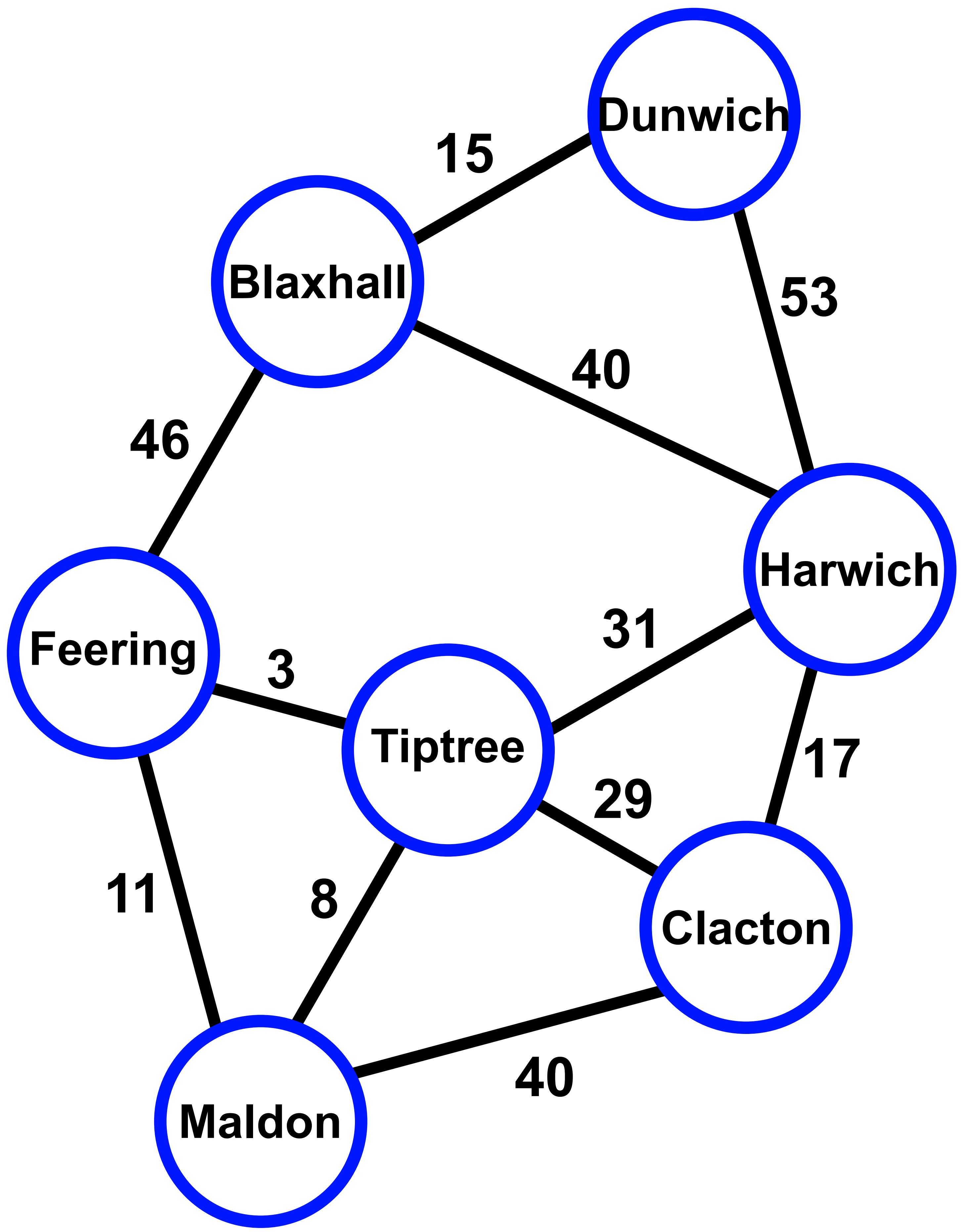

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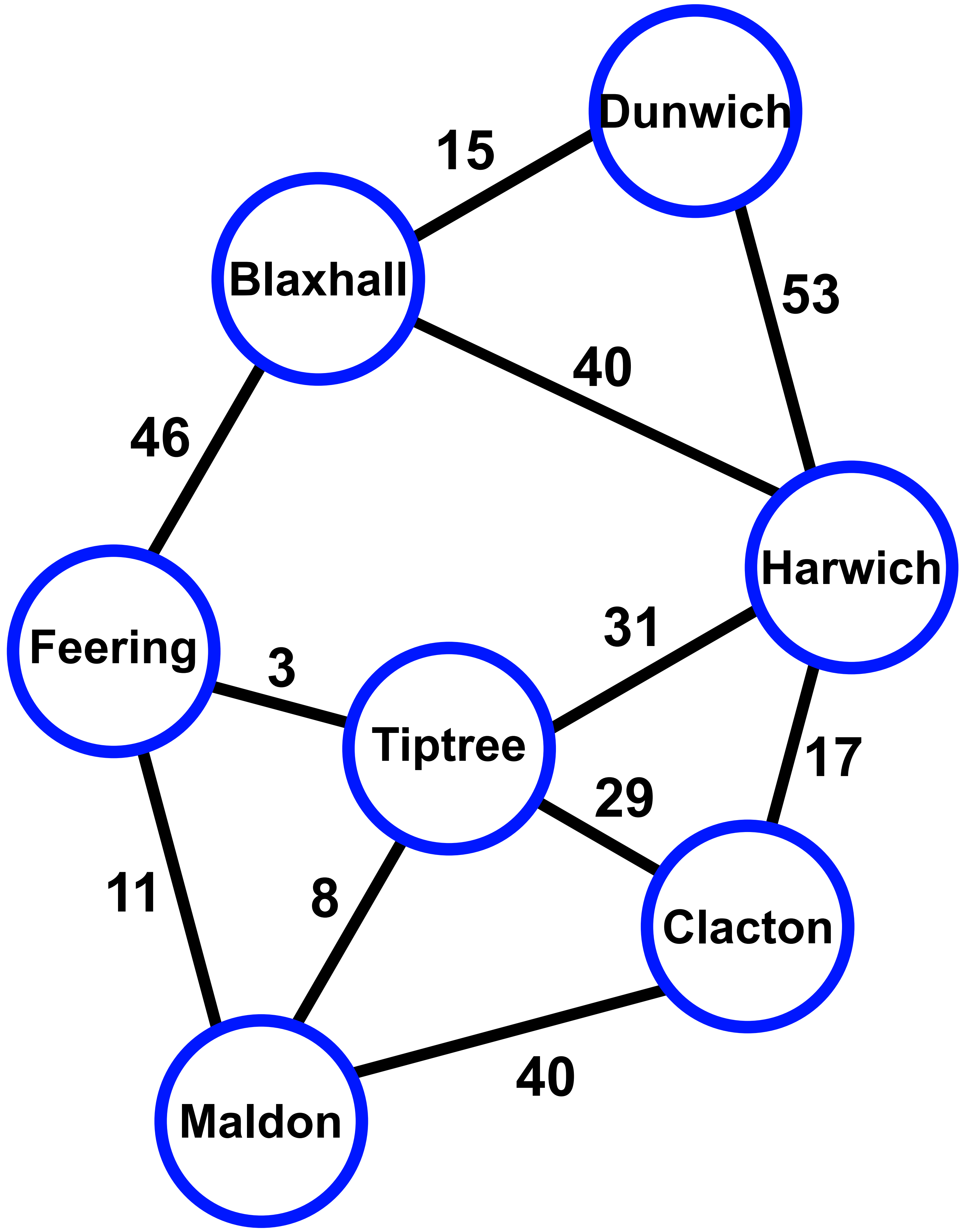
Vertices = towns

What is a graph?

Example



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Vertices = towns

Edges = town connections

What is a graph?

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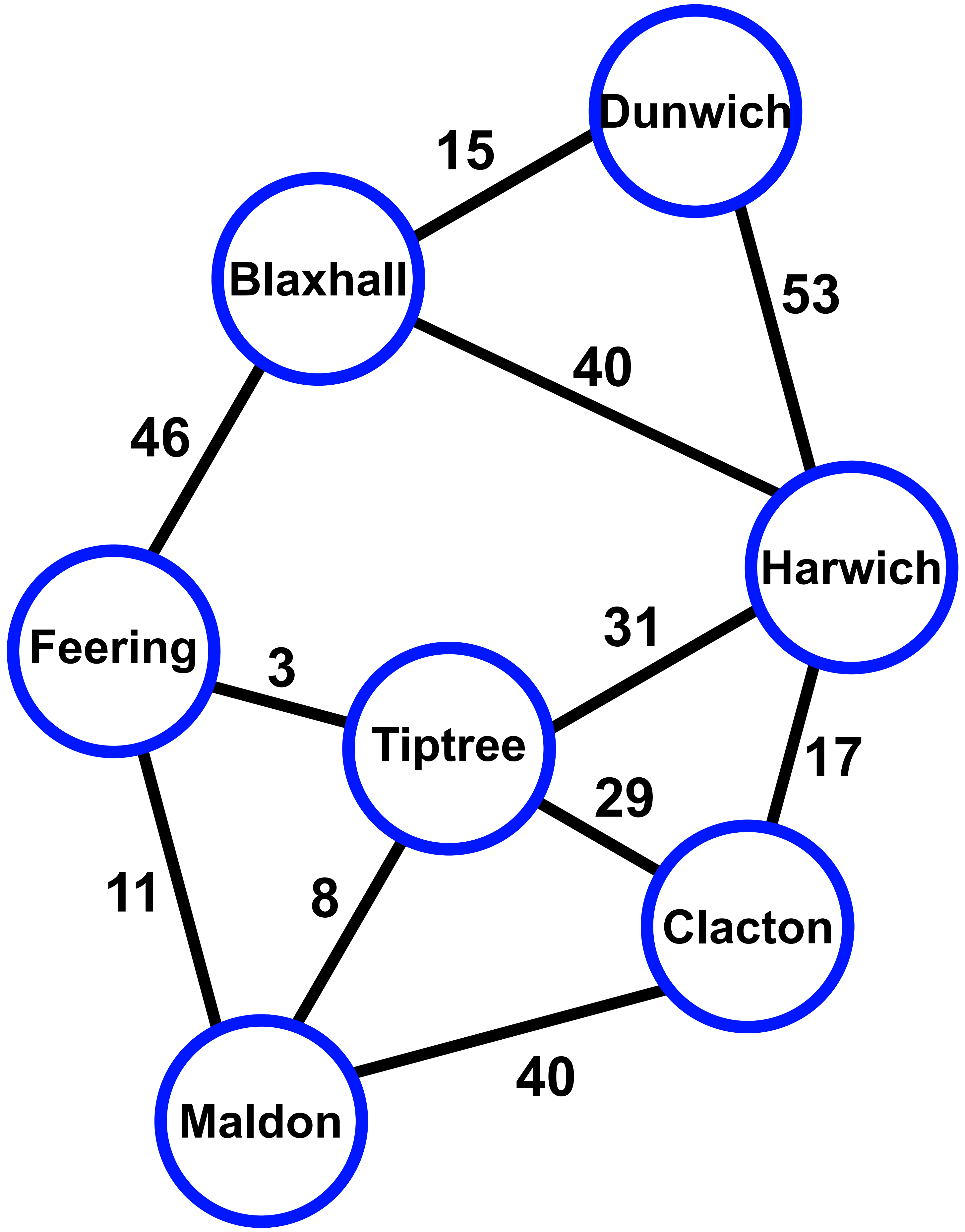

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Vertices = towns

Edges = town connections

Weights = distances between towns

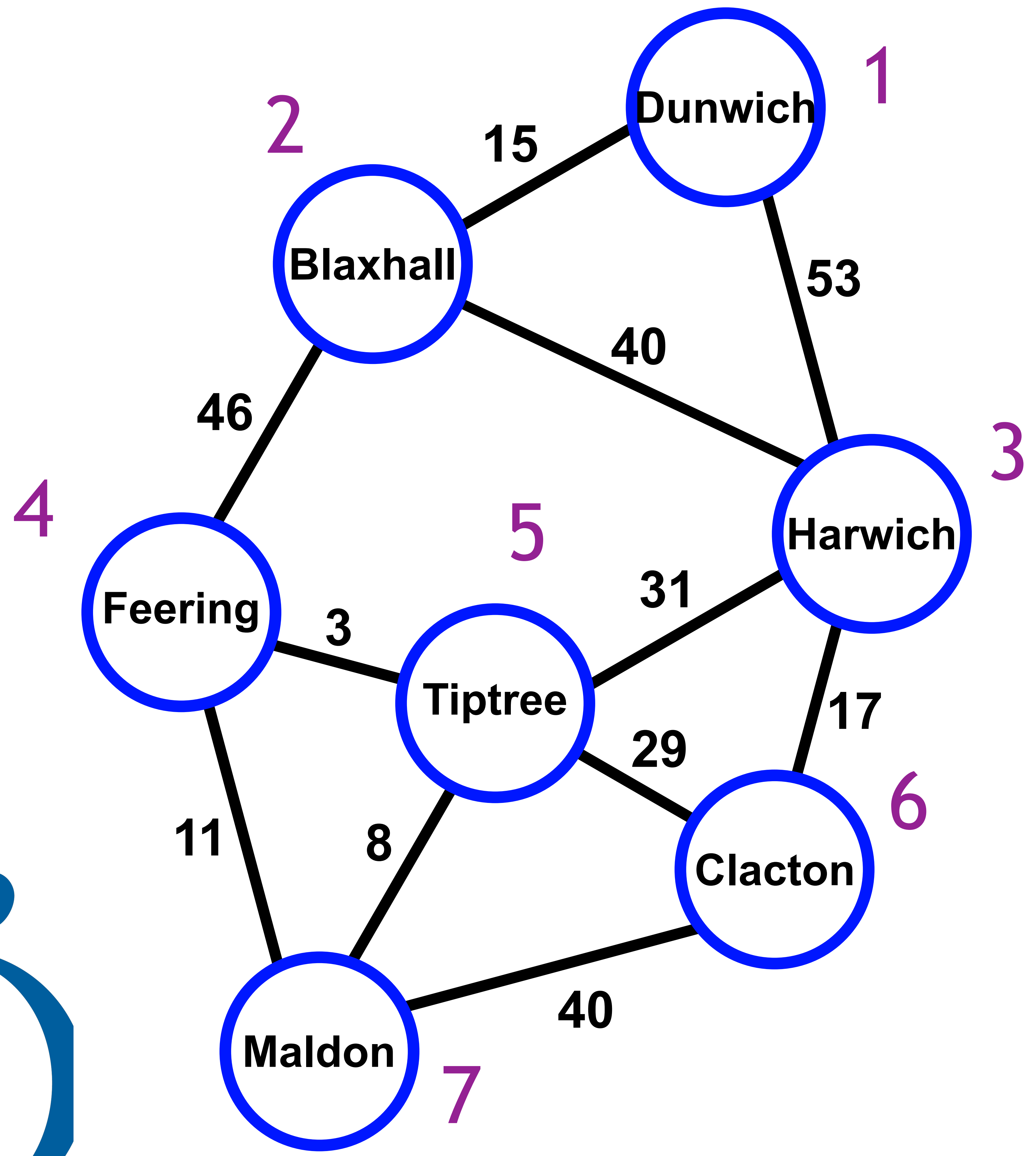
Incidence matrix

For a (weighted) graph (with weights w) we define a so-called *incidence matrix* $M_w \in \mathbb{R}^{|E|\times|V|}$, where $|E|$ denotes the number of edges and $|V|$ the number of vertices, as

$$(M_w)_{ev} := \begin{cases} \sqrt{w_{ev}} & \text{if } v = i \\ -\sqrt{w_{ev}} & \text{if } v = j \\ 0 & \text{otherwise} \end{cases},$$

where every edge $e = (i, j)$ connects vertices i and j, with $i > j$.

Incidence matrix

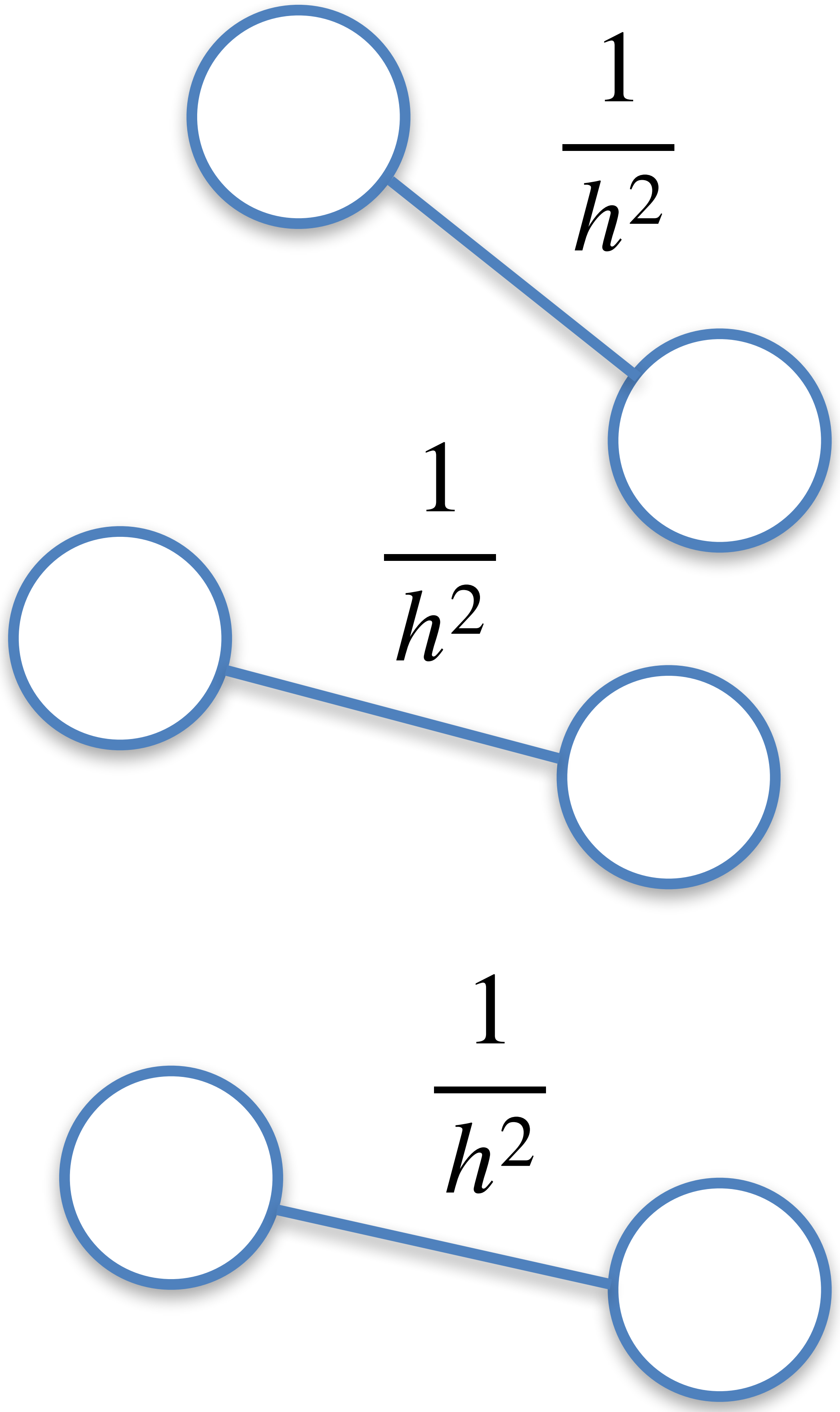


$$
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}
$$

$$
M_w = \begin{pmatrix}
-\sqrt{15} & \sqrt{15} & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{53} & 0 & \sqrt{53} & 0 & 0 & 0 & 0 \\
0 & -\sqrt{40} & \sqrt{40} & 0 & 0 & 0 & 0 \\
0 & -\sqrt{46} & 0 & 0 & \sqrt{46} & 0 & 0 \\
0 & 0 & 0 & -\sqrt{3} & \sqrt{3} & 0 & 0 \\
0 & 0 & -\sqrt{31} & \sqrt{31} & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{29} & 0 & \sqrt{29} & 0 \\
0 & 0 & -\sqrt{17} & 0 & 0 & \sqrt{17} & 0 \\
0 & 0 & 0 & 0 & -\sqrt{11} & 0 & \sqrt{11} \\
0 & 0 & 0 & -\sqrt{8} & 0 & 0 & \sqrt{8} \\
0 & 0 & 0 & 0 & 0 & -\sqrt{40} & \sqrt{40}
\end{pmatrix}
\begin{array}{l} E1 \\ E2 \\ E3 \\ E4 \\ E5 \\ E6 \\ E6 \\ E7 \\ E8 \\ E9 \\ E10 \end{array}
$$

What is a graph?

Another example: finite differences



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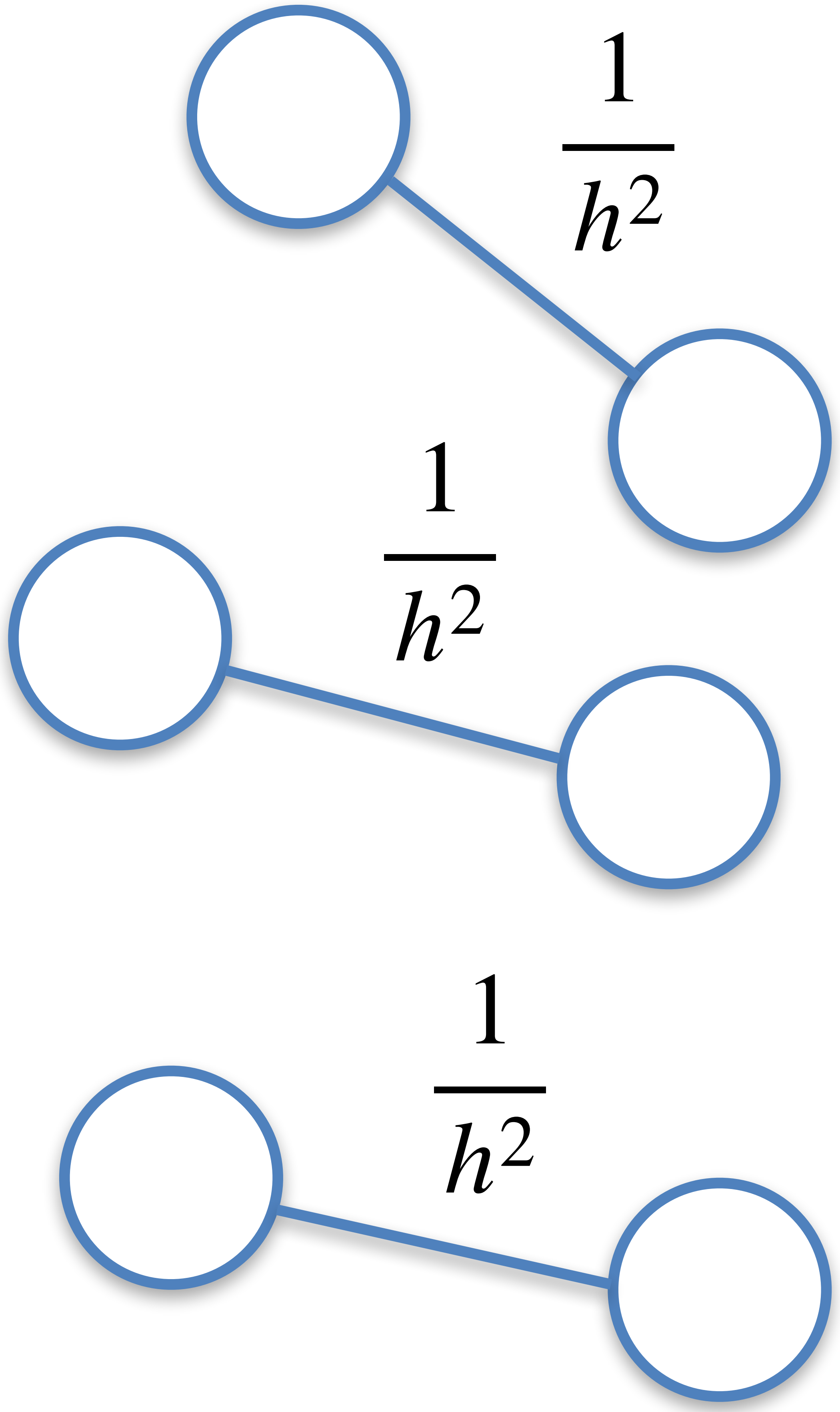

Every vertex is connected only to one other vertex

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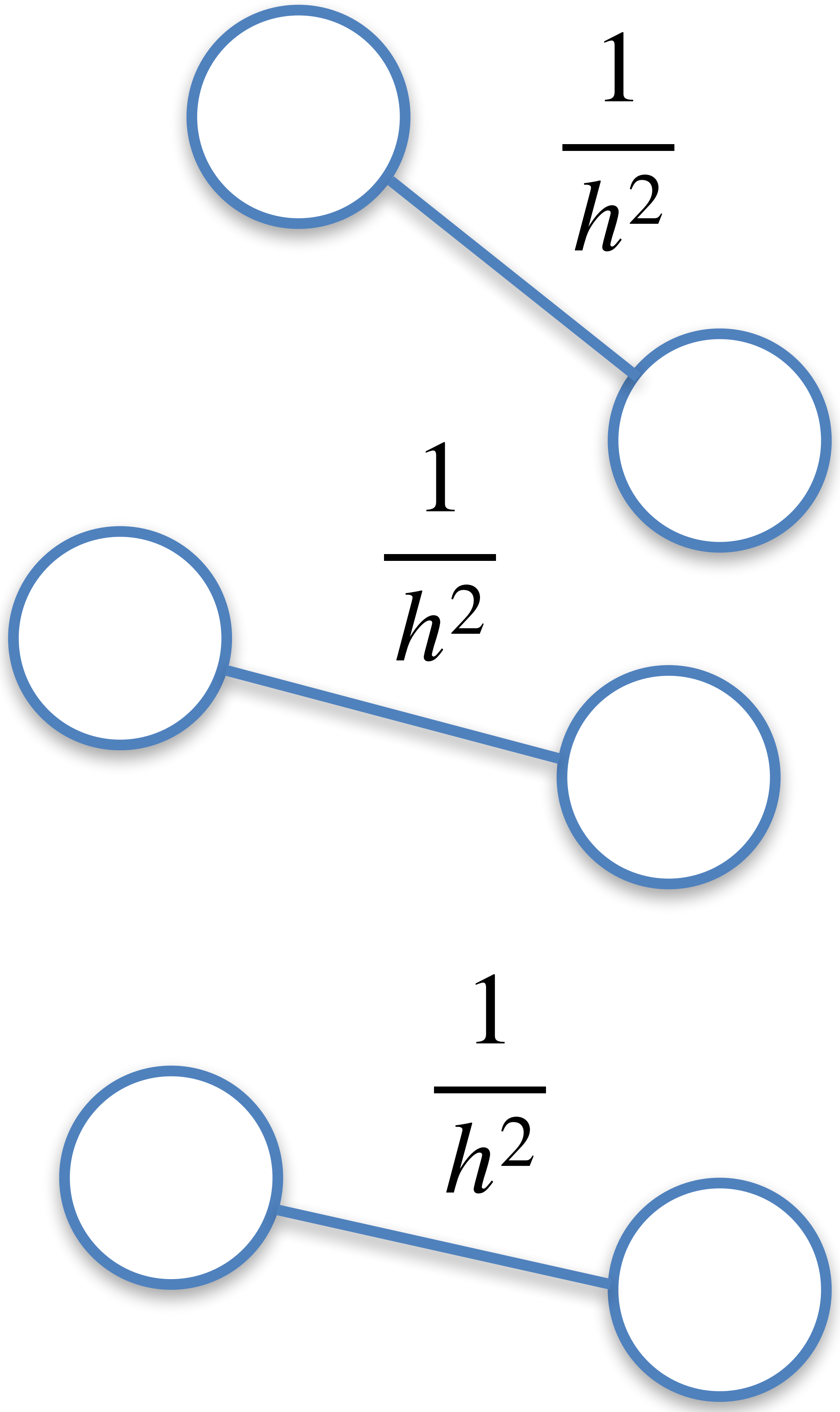

Every vertex is connected only to one other vertex

The weight is a constant factor $(1/h)^2$

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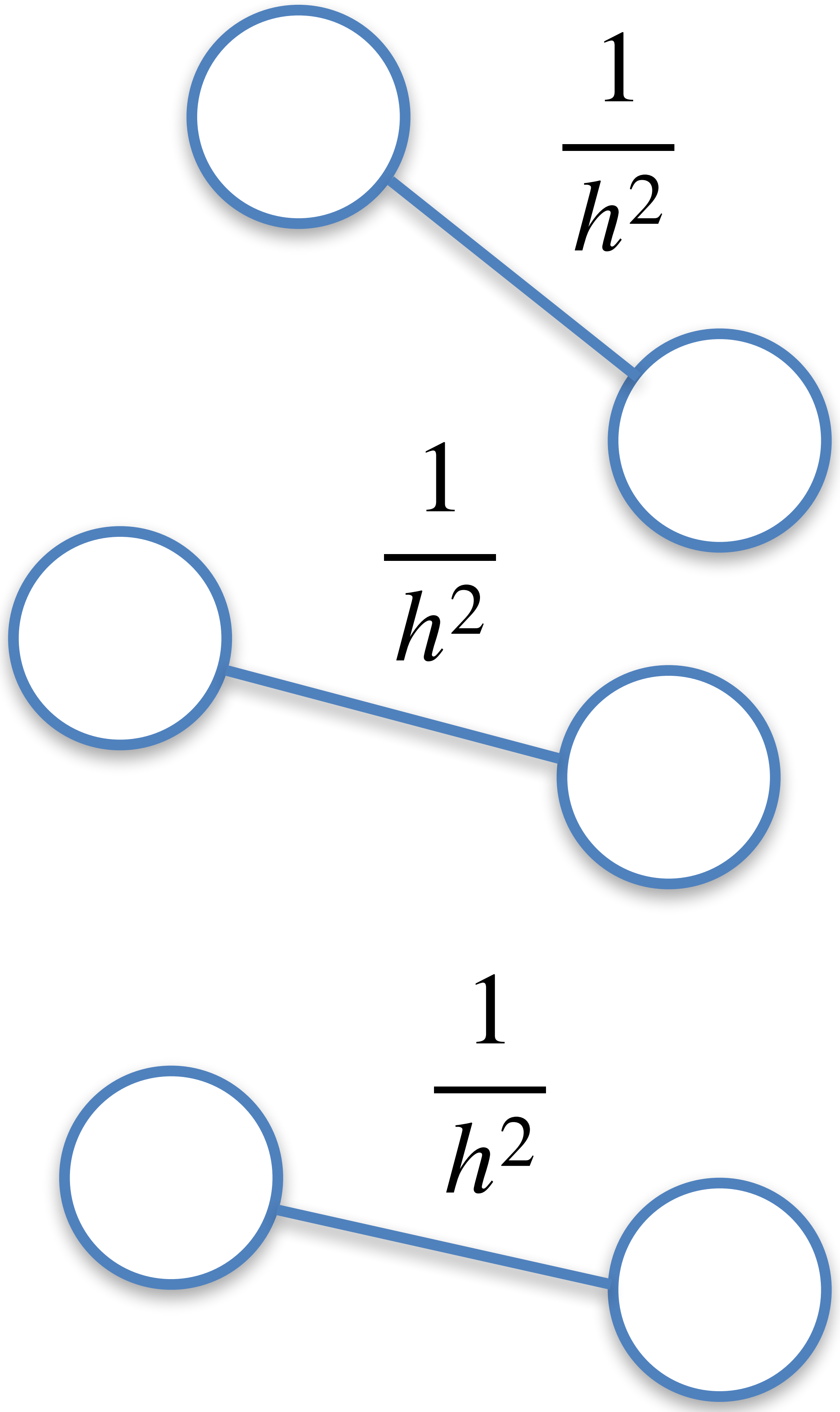

Every vertex is connected only to one other vertex

The weight is a constant factor $(1/h)^2$

Suppose every vertex represents $f(x_i)$:

What is a graph?

$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}$ can be written as matrix-vector multiplication

$$\begin{pmatrix} f'(x_1) \\ f'(x_2) \\ \vdots \\ f'(x_d) \end{pmatrix} \approx \frac{1}{h} \begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{d+1}) \end{pmatrix}$$

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This is our *incidence matrix*

Graph Laplacian

Based on the finite difference approximation

$$M_{\frac{1}{h}} = \frac{1}{h} \begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix},$$

it is natural to define second-order finite differences (or *Laplacians* in higher dimensions) as

$$L_{\frac{1}{h}} = M_{\frac{1}{h}}^{\top} M_{\frac{1}{h}}$$

Graph Laplacian

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it is natural to define second-order finite differences (or *Laplacians* in higher dimensions) as

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We can define the same for arbitrary graphs!

Graph Laplacian

The *graph-Laplacian* $L_w \in \mathbb{R}^{|V| \times |V|}$ is defined as

$$L_w := M_w^\top M_w \, .$$

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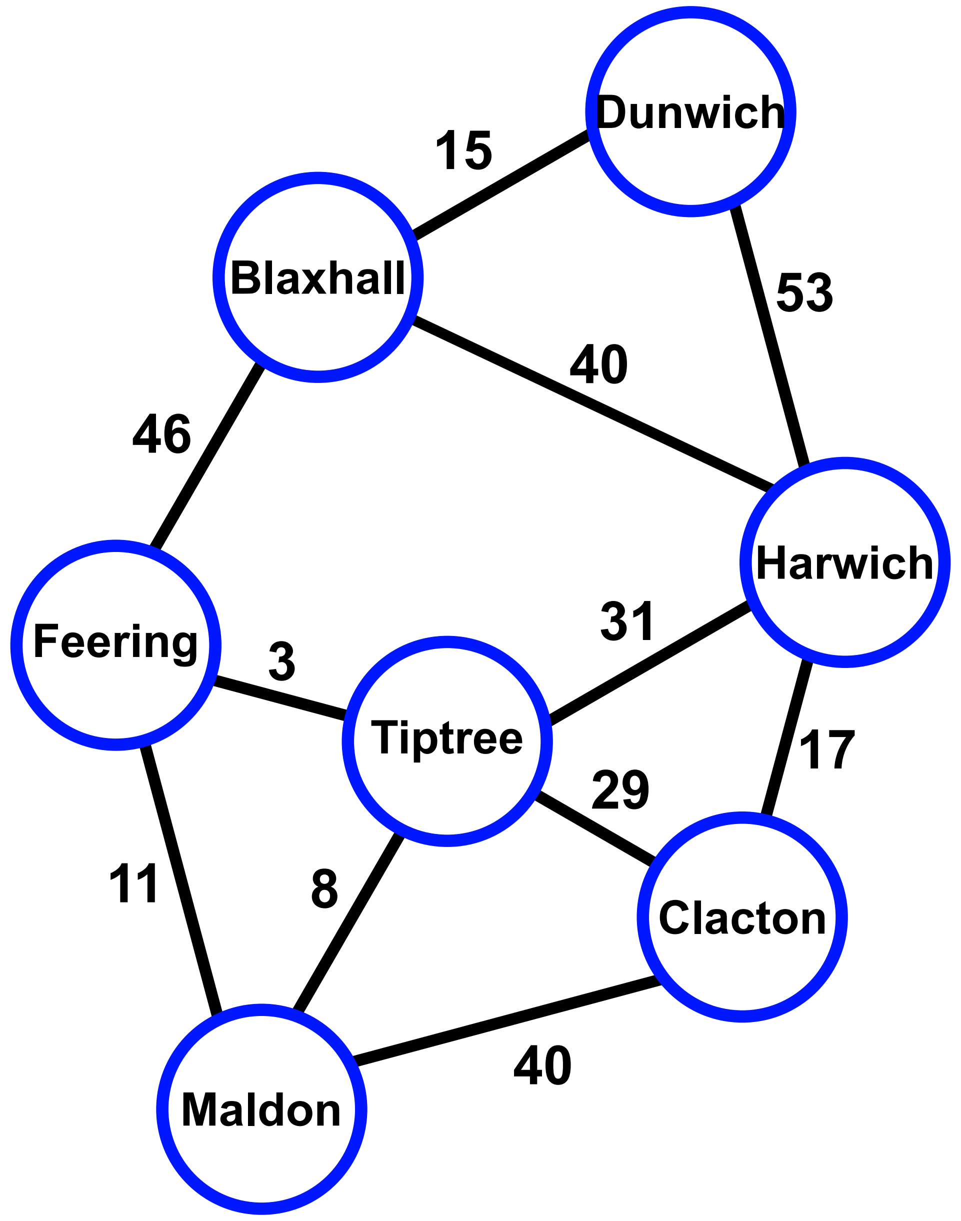


$$L_w = M_w^\top M_w = \begin{pmatrix} 68 & -15 & -53 & 0 & 0 & 0 & 0 \\ -15 & 101 & -40 & 0 & -46 & 0 & 0 \\ -53 & -40 & 141 & -31 & 0 & -17 & 0 \\ 0 & 0 & -31 & 71 & -3 & -29 & -8 \\ 0 & -46 & 0 & -3 & 60 & 0 & -11 \\ 0 & 0 & -17 & -29 & 0 & 86 & -40 \\ 0 & 0 & 0 & -8 & -11 & -40 & 59 \end{pmatrix}$$

Graph Laplacian

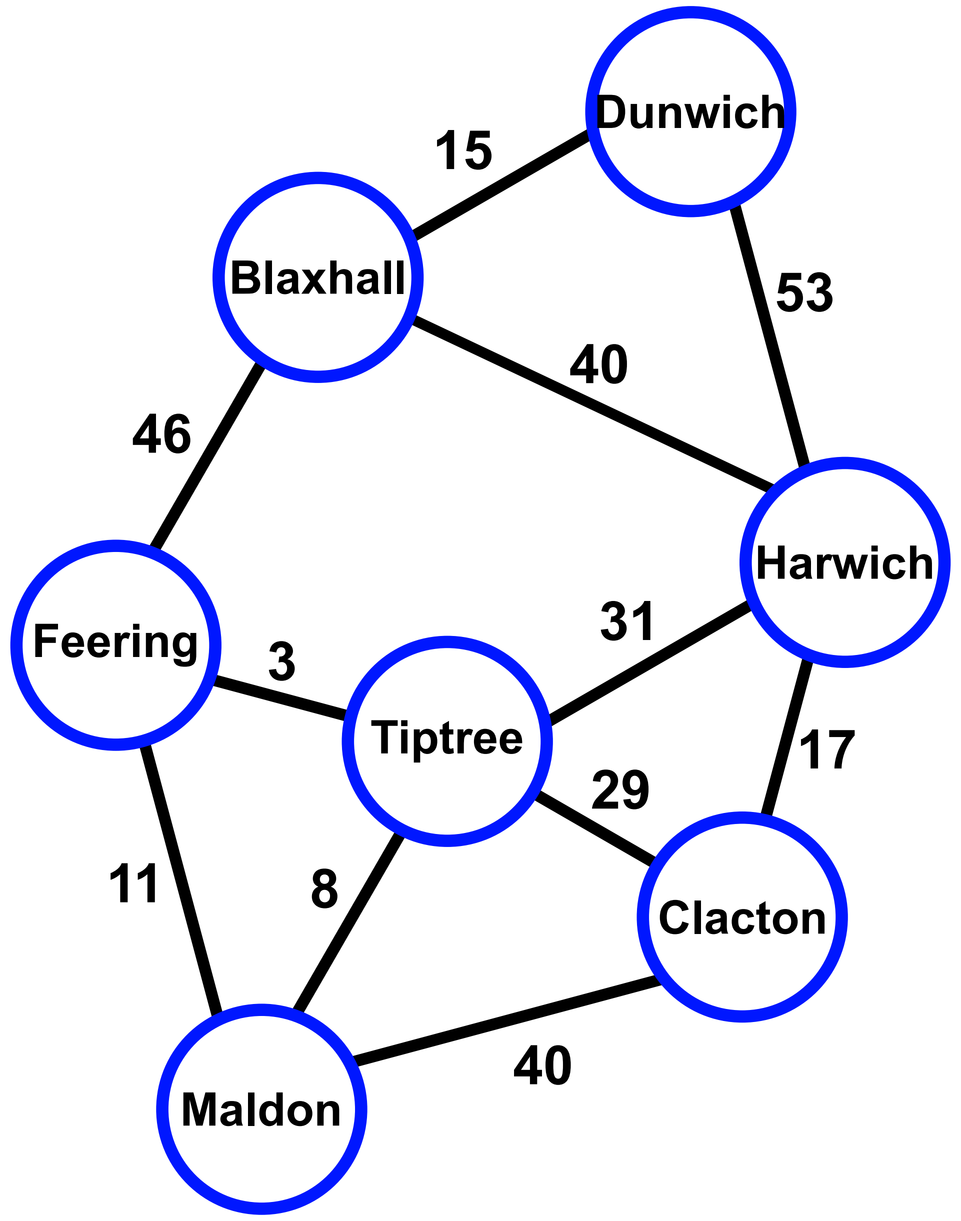
The *graph-Laplacian* $L_w \in \mathbb{R}^{|V|\times|V|}$ is defined also as

Degree matrix

Adjacency matrix

Graph Laplacian

The *graph-Laplacian* $L_w \in \mathbb{R}^{|V|\times|V|}$ is defined also as $L_w = D_w - A_w$



$$D = \begin{pmatrix} 68 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 101 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 141 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 71 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 60 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 86 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 59 \end{pmatrix}$$

Degree matrix

$$A = \begin{pmatrix} 0 & 15 & 53 & 0 & 0 & 0 & 0 \\ 15 & 0 & 40 & 0 & 46 & 0 & 0 \\ 53 & 40 & 0 & 31 & 0 & 17 & 0 \\ 0 & 0 & 31 & 0 & 3 & 29 & 8 \\ 0 & 46 & 0 & 3 & 0 & 0 & 11 \\ 0 & 0 & 17 & 29 & 0 & 0 & 40 \\ 0 & 0 & 0 & 8 & 11 & 40 & 0 \end{pmatrix}$$

Adjacency matrix

Semi-supervised learning

We can use incidence matrices and graph-Laplacians to model and exploit similarities in a dataset

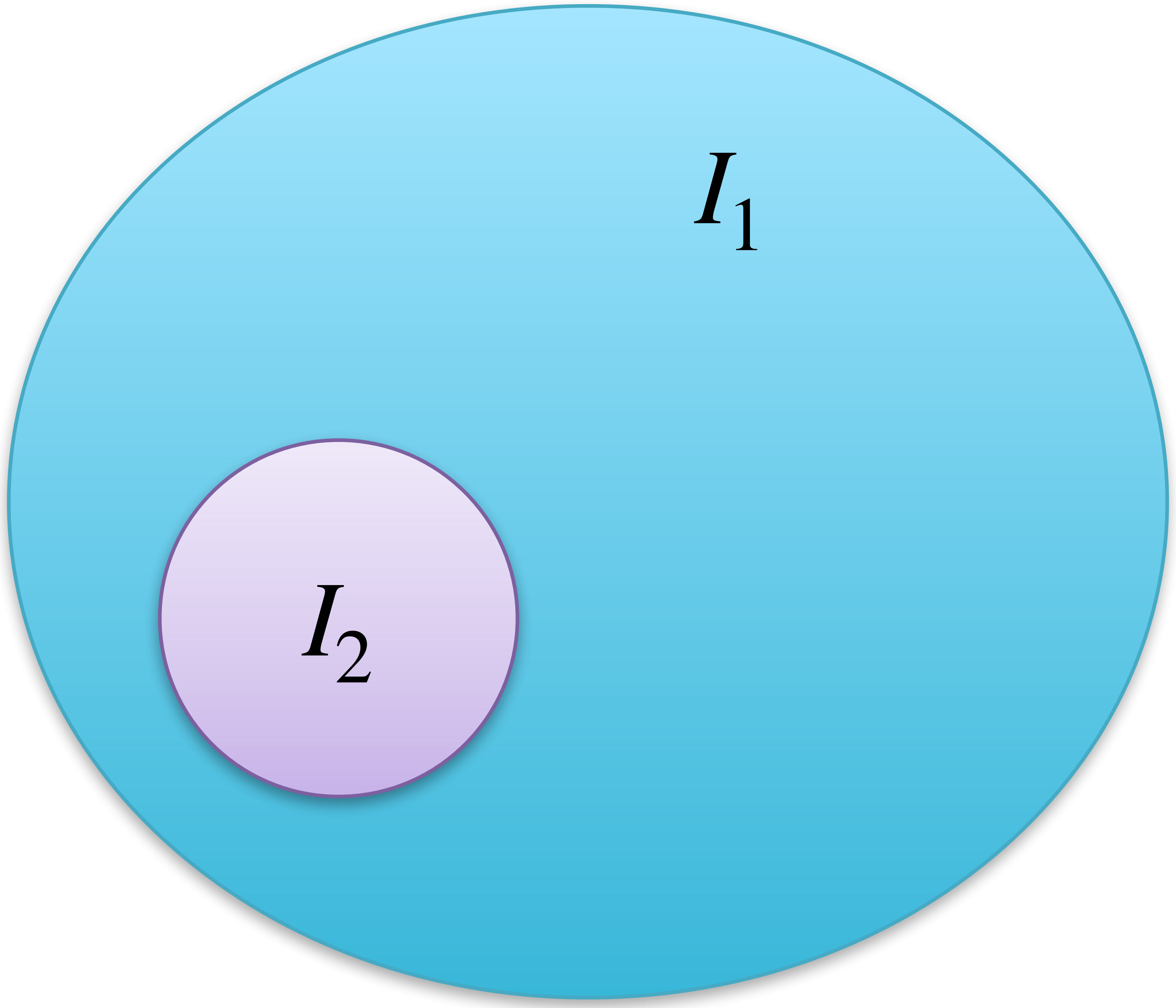
Interpolation

Suppose we are given data points $\{x_i\}_{i\in I_1}$ and pairs $\{(x_j, y_j)\}_{j\in I_2}$ with $I_2 \subset I_1$;

How do we find $\{(x_i, y_i)\}_{i\in I_1}$?

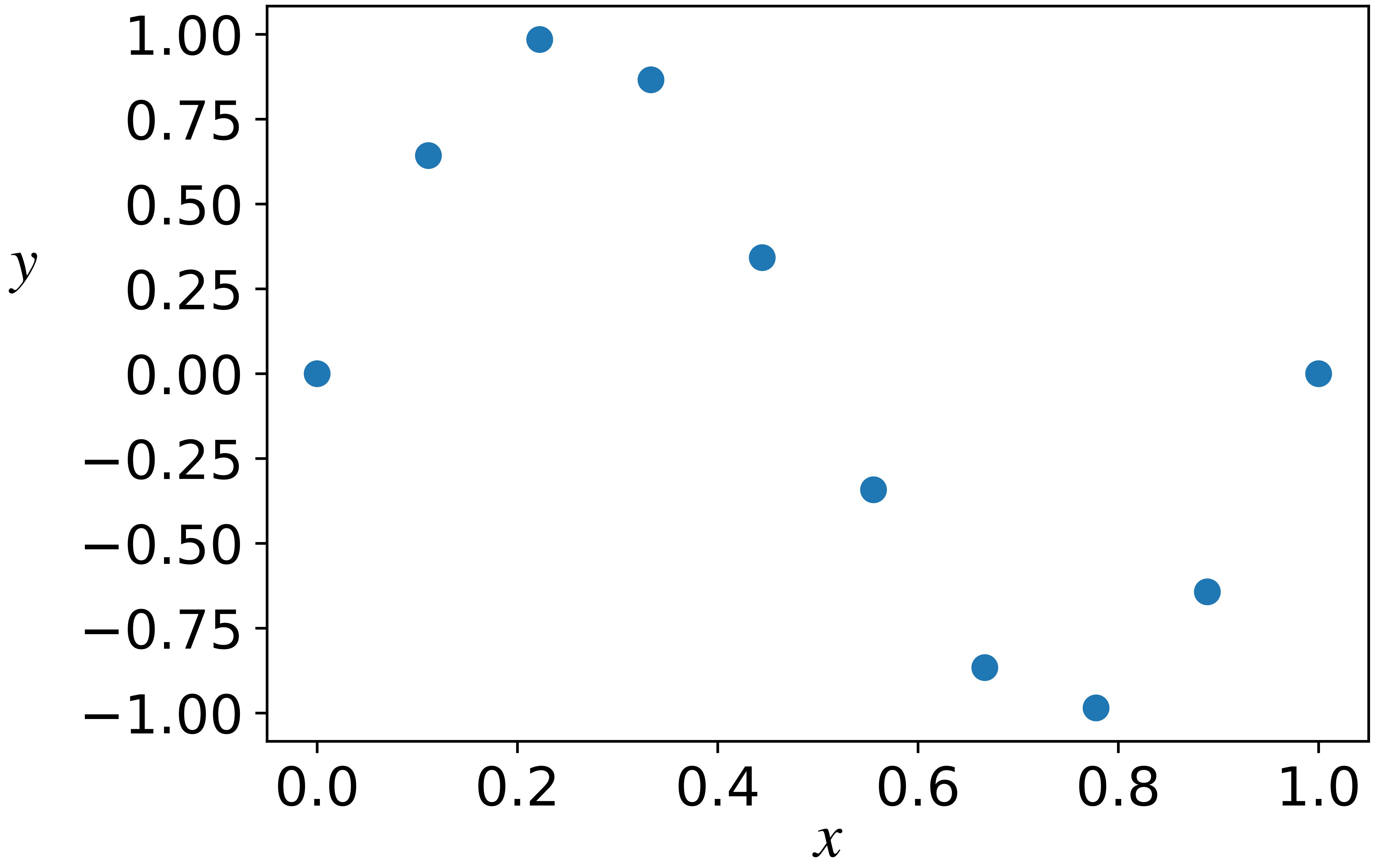
Interpolation

For each x_i in I_2 we know the correspondent y_i



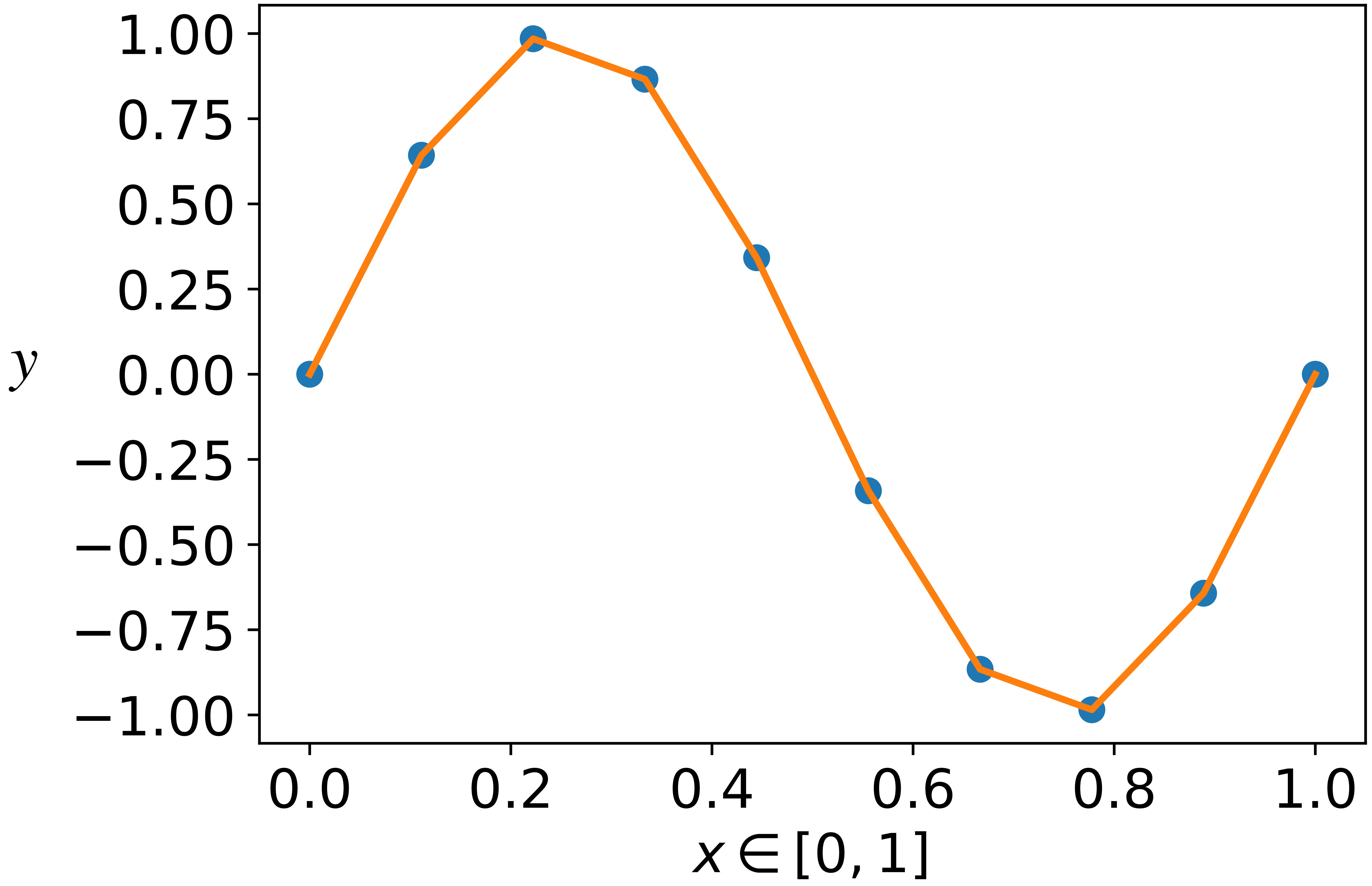
Interpolation

In general we don’t know the underling function, how can we connect the dots?



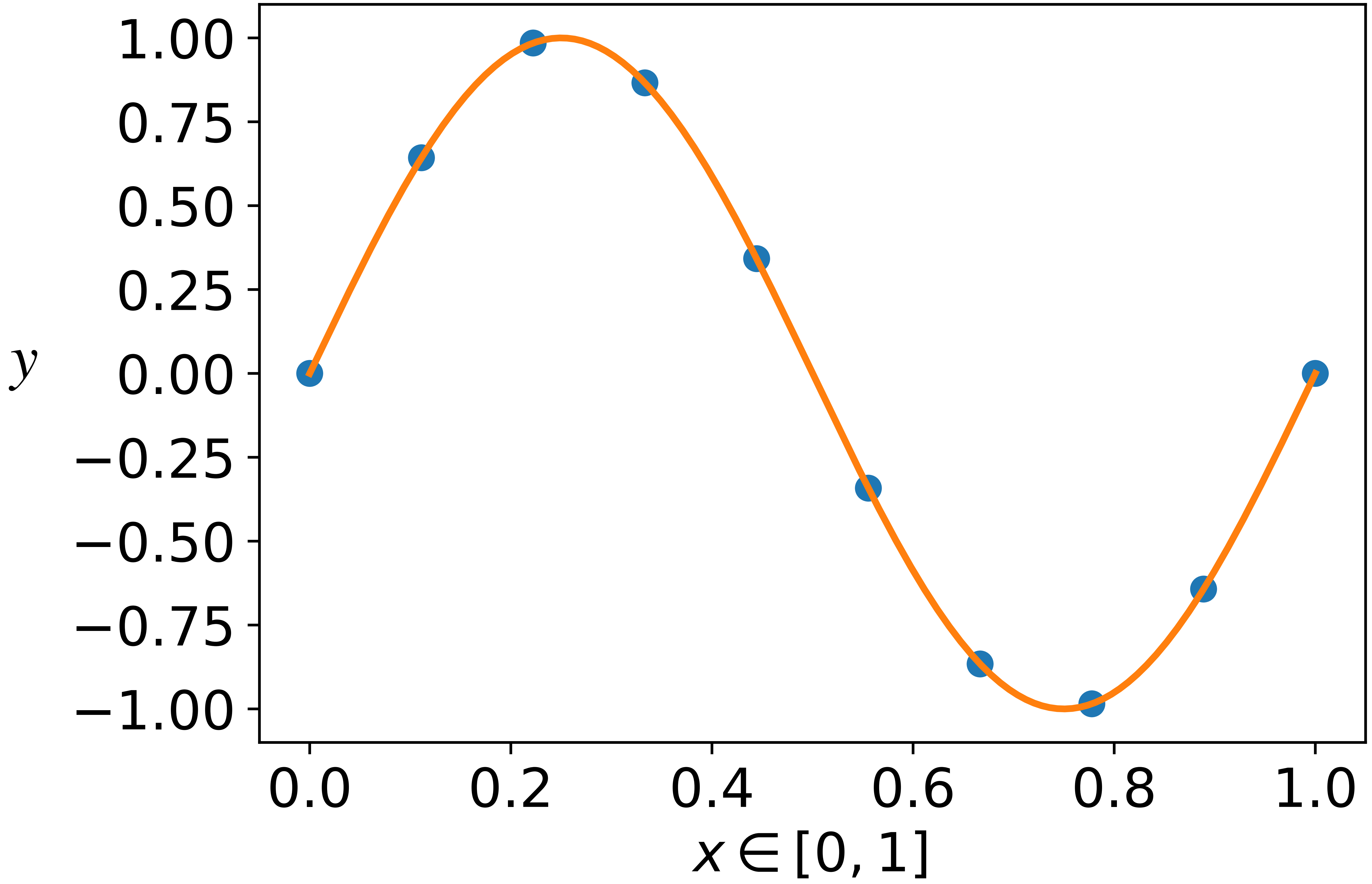
Interpolation

Linear interpolation



Interpolation

Smoother interpolation



Interpolation

One way of formulating this problem mathematically uses the ideas of optimization:

$$\min_{\{y_i\}_{i \in I_1}} E(\mathbf{y}) \qquad \text{subject to} \qquad (\mathbf{P}_{I_2}\mathbf{y})_j = y_j \quad \forall j \in I_2$$

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How to choose E to interpolate?

Interpolation

We cannot use the MSE! Since, we miss the ground truth for the new $\mathbf{y}$

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Proposal:

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Proposal:

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$$E(\mathbf{y}) = \left\| \mathbf{M}_{\frac{1}{h}} \mathbf{y} \right\|^2 \qquad \text{with} \qquad \mathbf{M}_{\frac{1}{h}} = \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix},$$

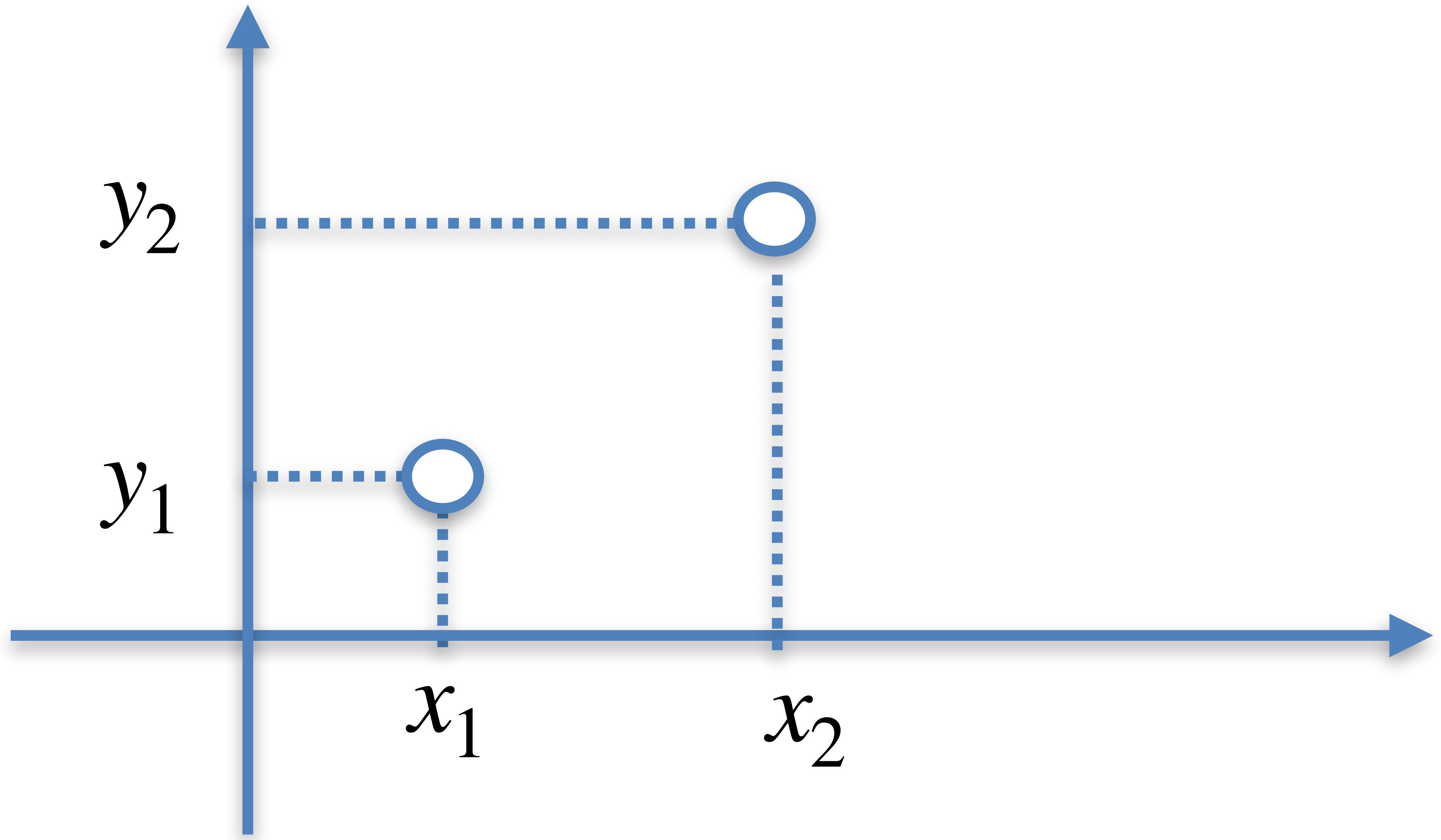
Interpolation

Interpolation

A simple example might help us understand why this is a good idea

Interpolation

Imagine that we are given these two points



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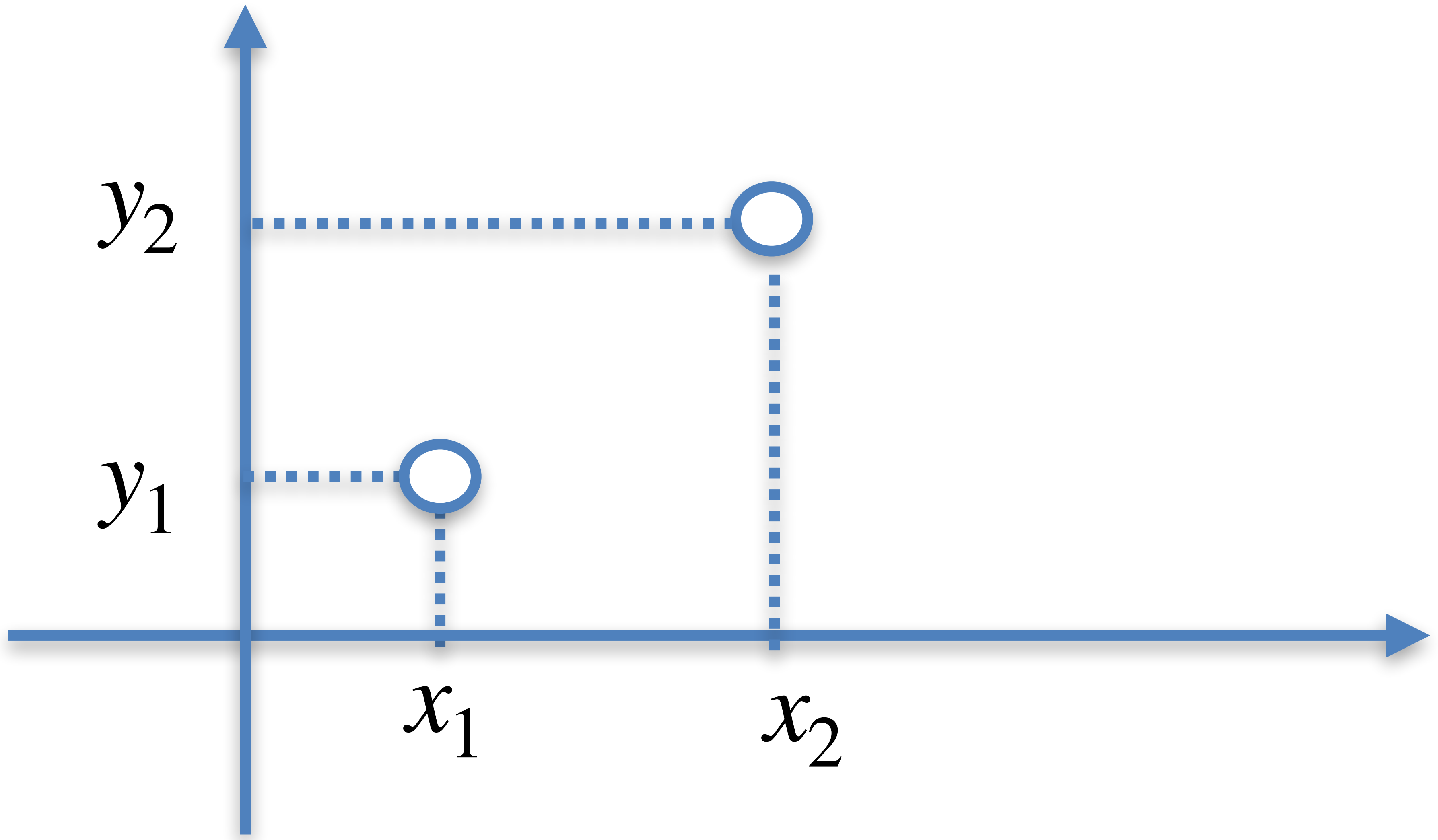

We would add another point, between x_1 and x_3, thus interpolating

Interpolation

The goal is to find y_3 (in interpolation x_3 is in the middle between the other two points)

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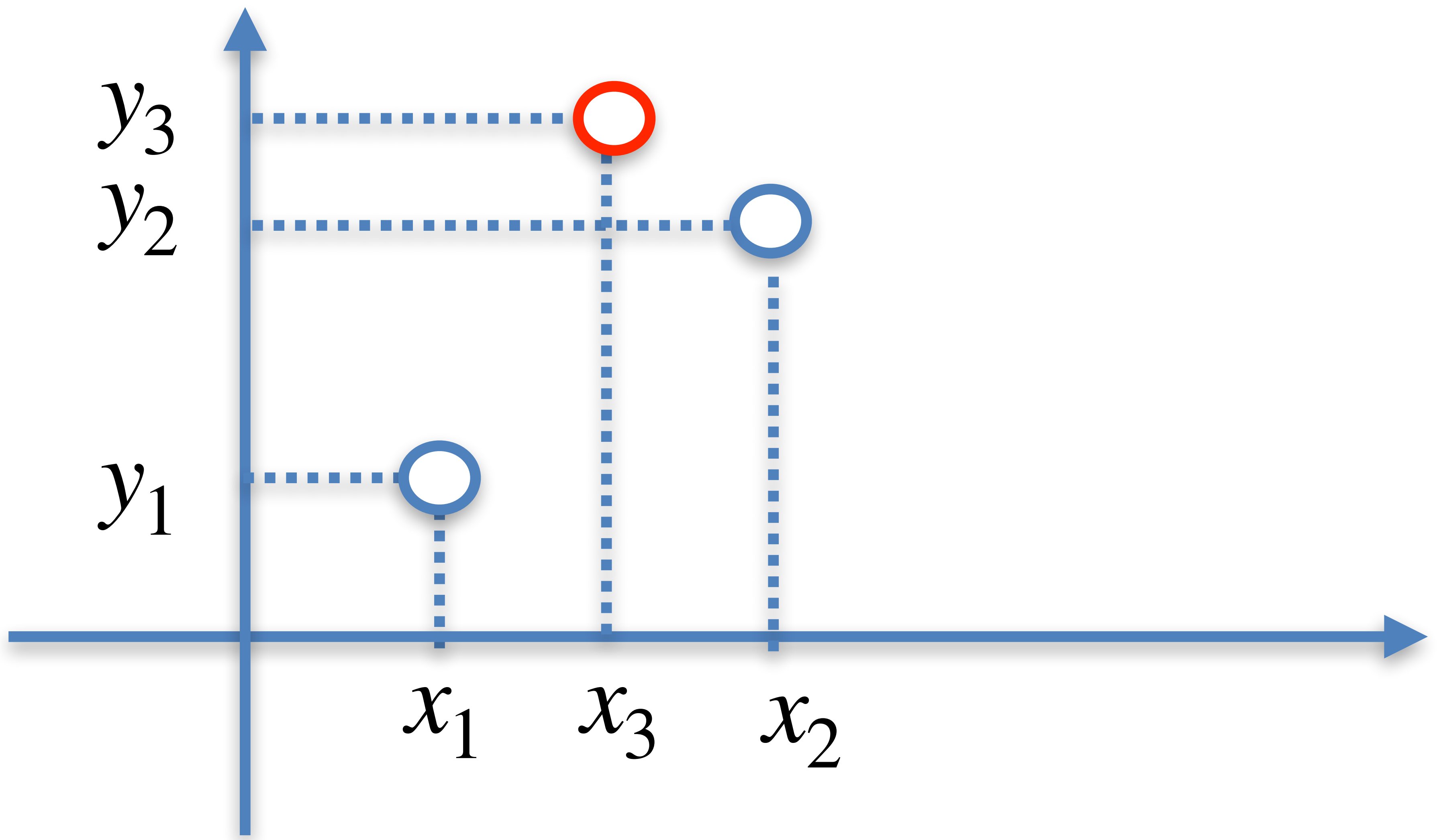

Is it here?

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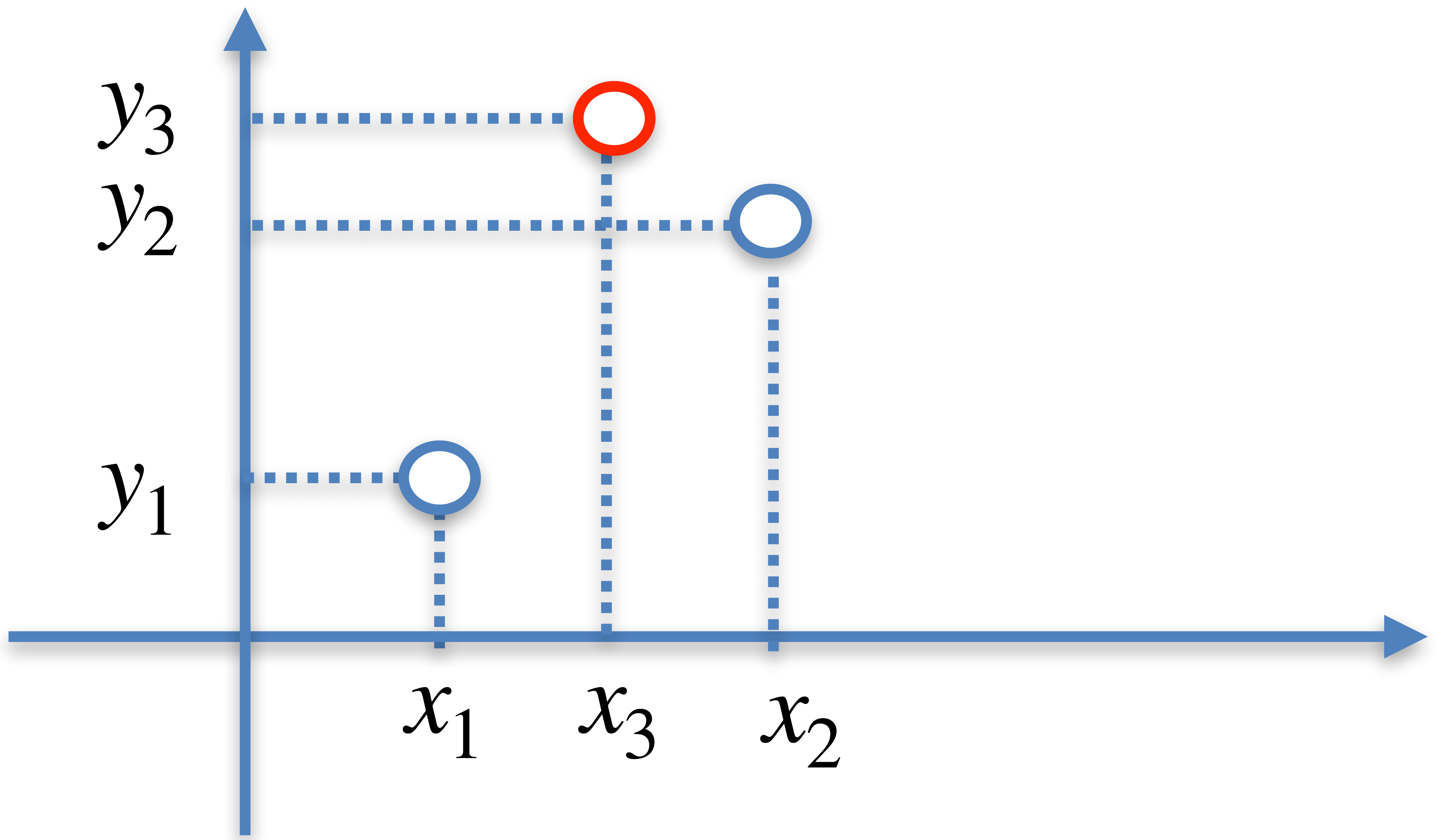

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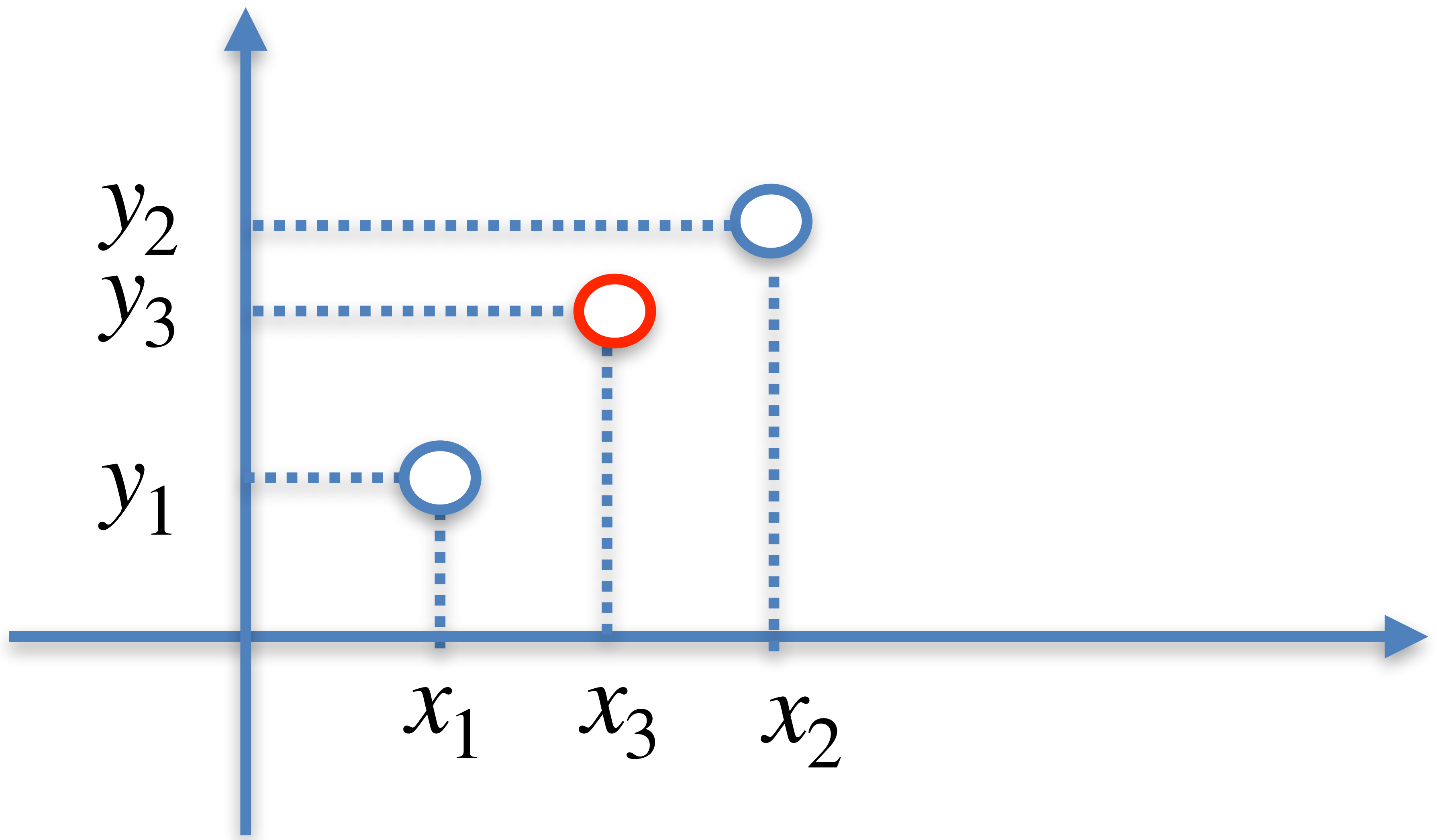

Or here?

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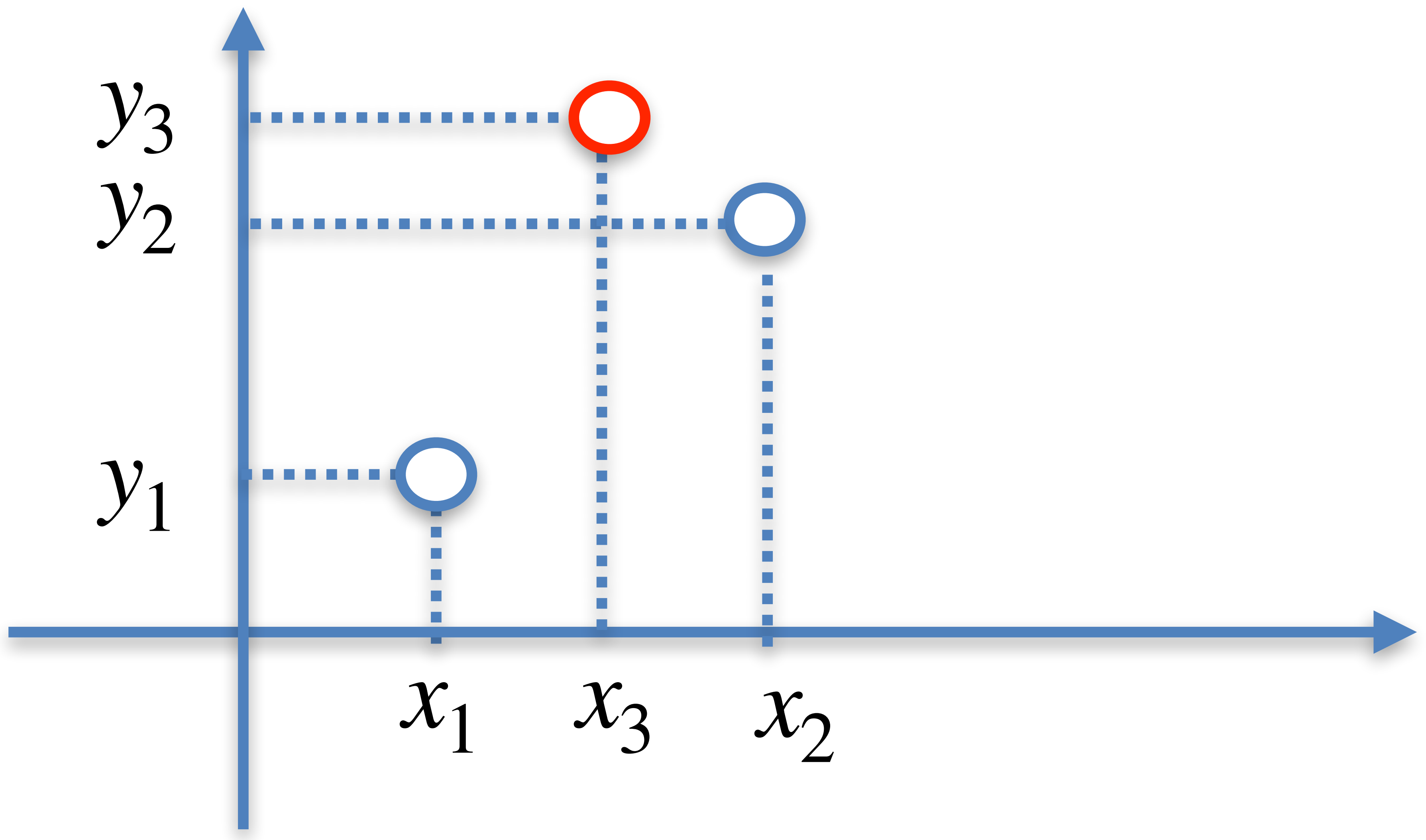

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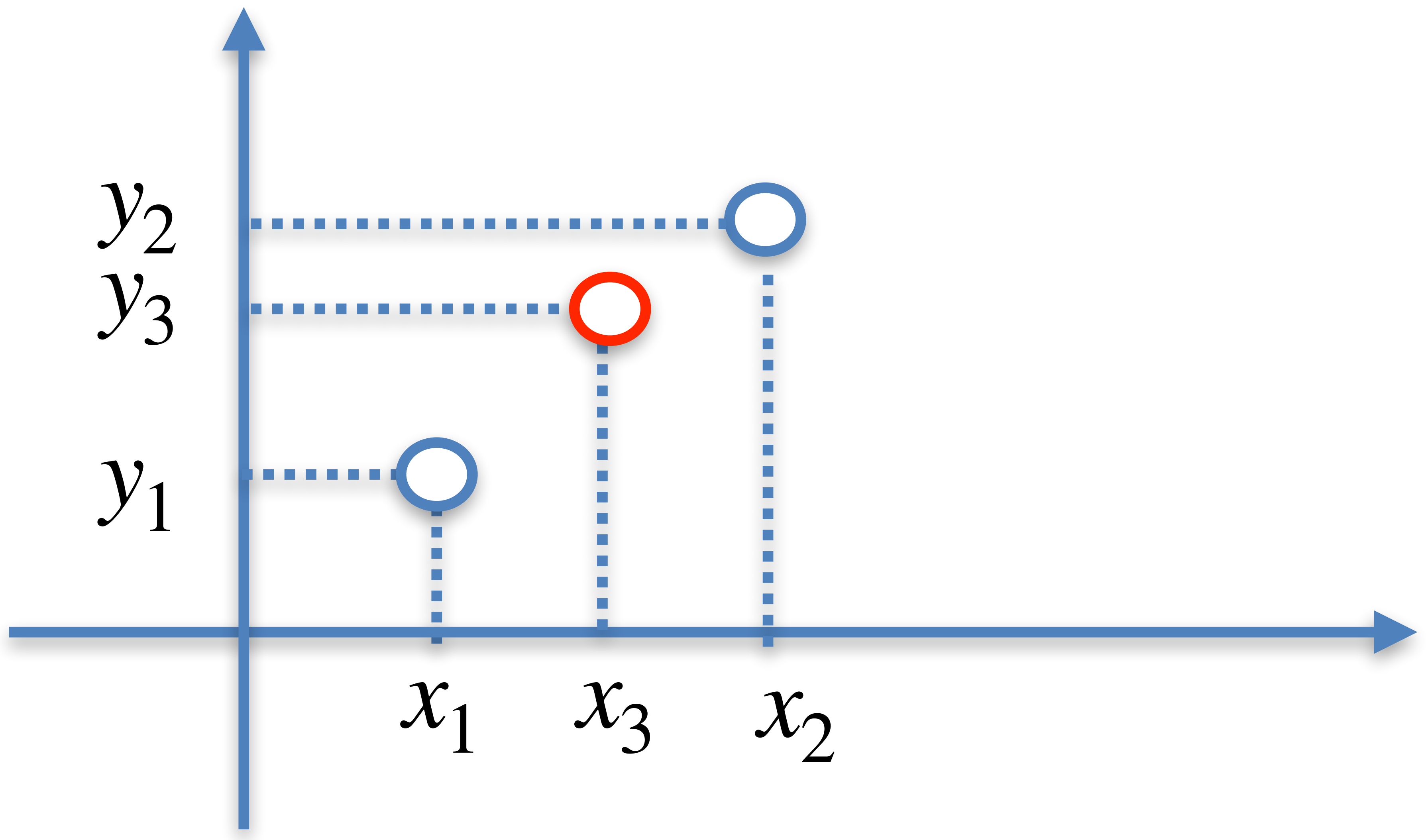

Or here?

Hence, $y_3 = ?$

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We can now see how using the incidence matrix and minimising E might help

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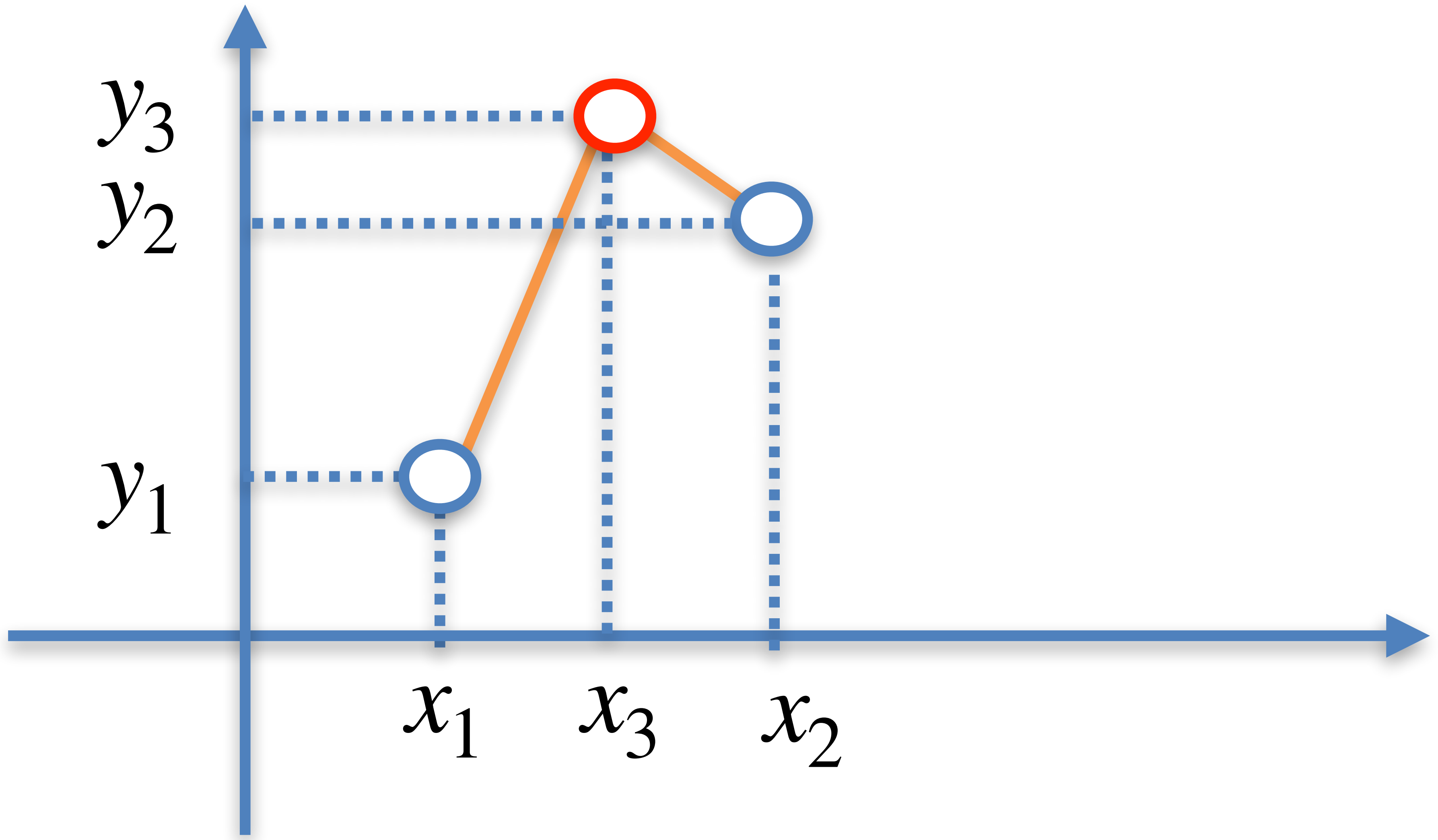

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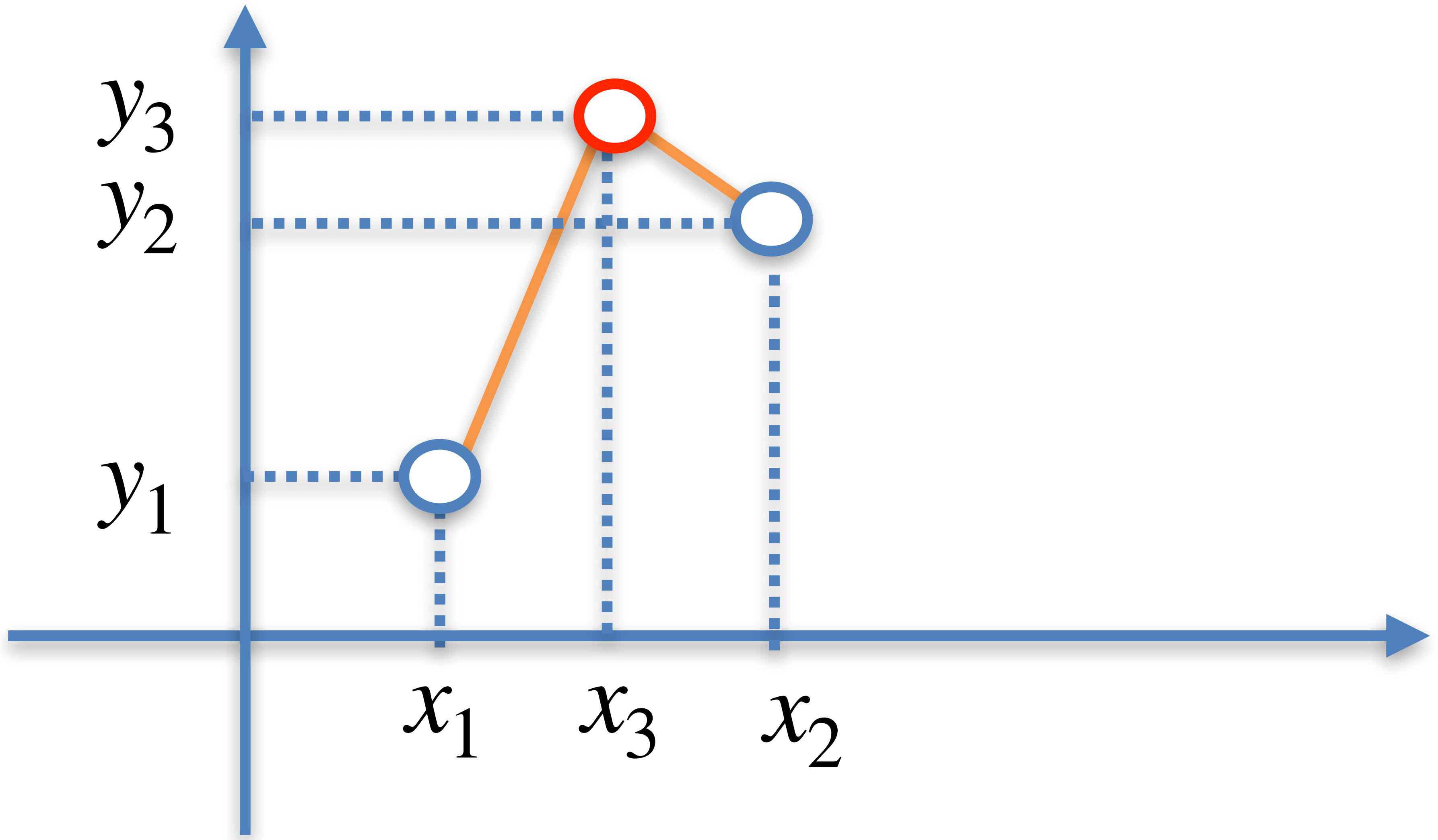

$$M_{\frac{1}{h}} = \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

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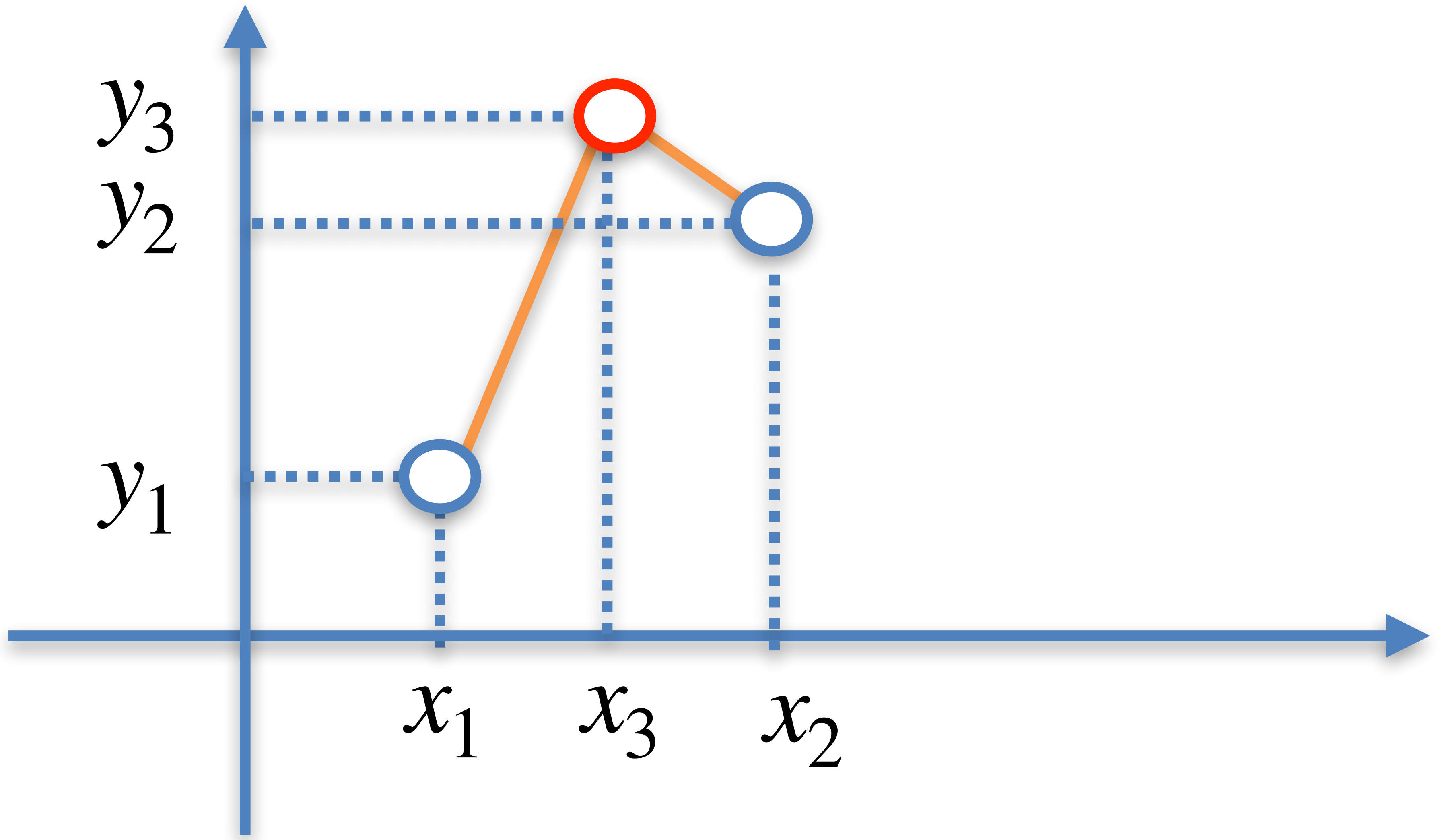

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What is $\mathbf{y}$?

Interpolation

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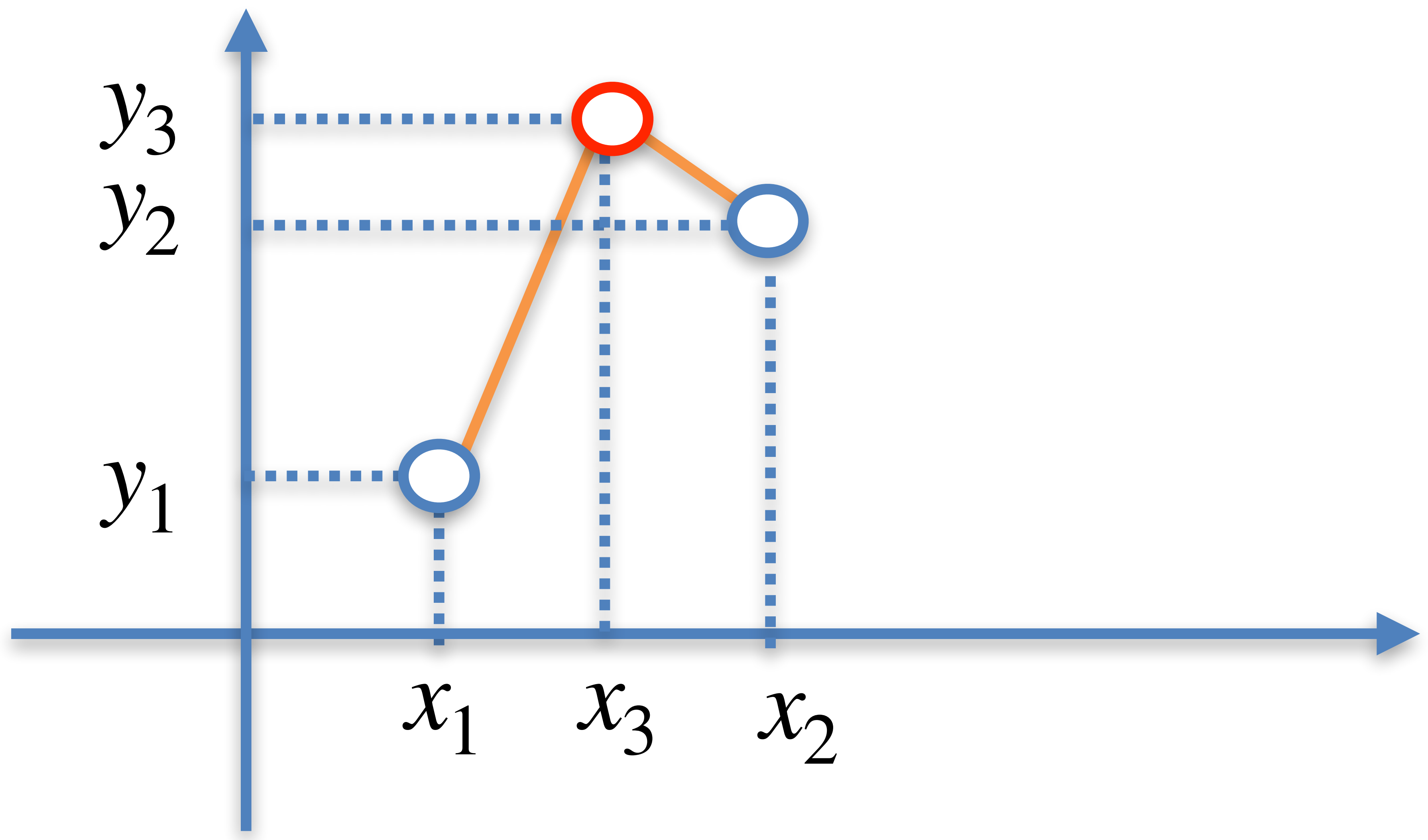

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What is **y** ?

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_3 \\ y_2 \end{pmatrix}$$

Interpolation

Given

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Interpolation

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$$M_{\frac{1}{h}} = \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_3 \\ y_2 \end{pmatrix}$$

Their product is

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Interpolation

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Hence, the proposed energy function becomes

Interpolation

Given

$$M_{\frac{1}{h}} = \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_3 \\ y_2 \end{pmatrix}$$

Their product is

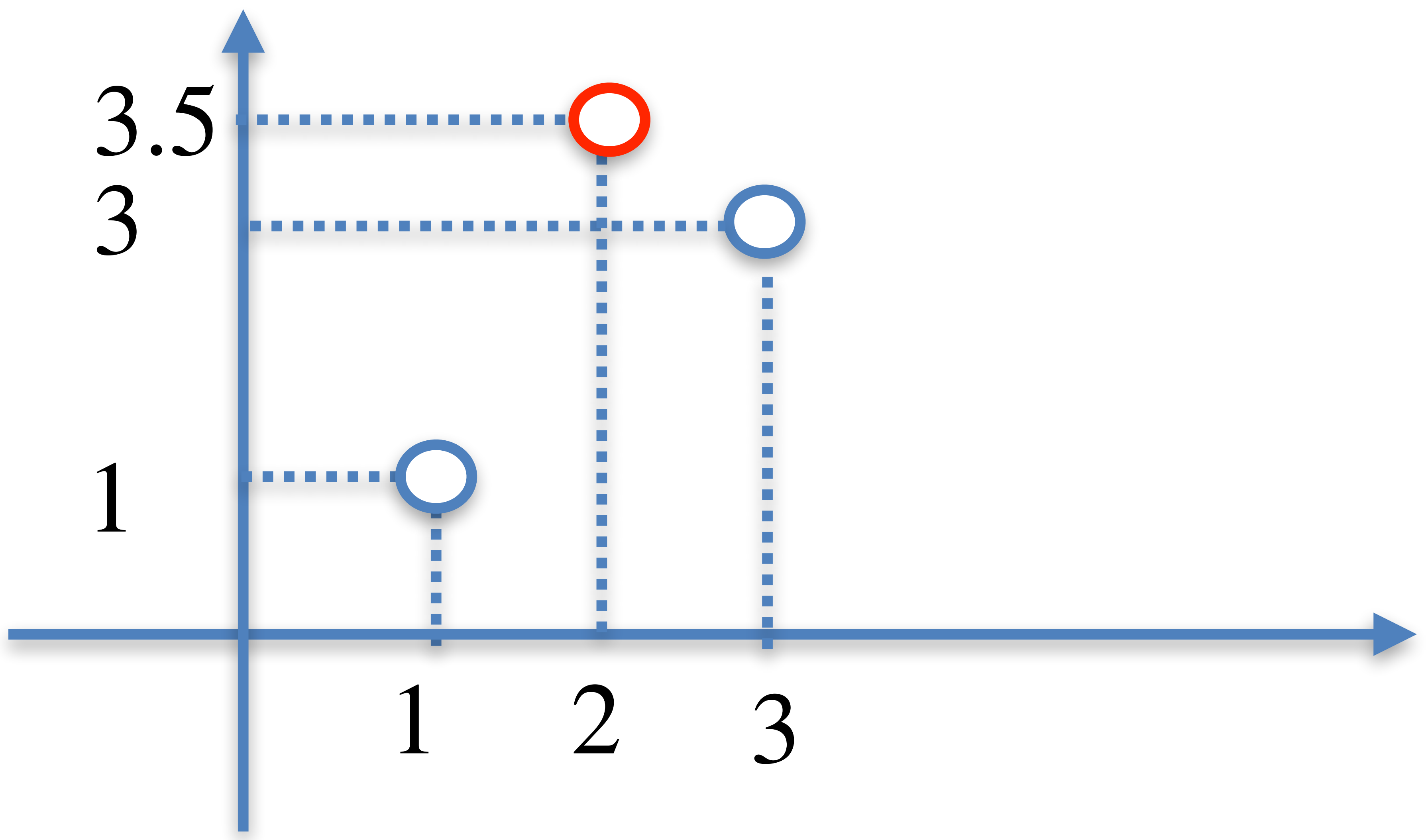
$$\mathbf{M}_{\frac{1}{h}}\mathbf{y} = \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_3 \\ y_2 \end{pmatrix} = \frac{1}{h}\begin{pmatrix} y_3 - y_1 \\ y_2 - y_3 \end{pmatrix}$$

Hence, the proposed energy function becomes

$$E(\mathbf{y}) = \left\| \mathbf{M}_{\frac{1}{h}}\,\mathbf{y} \right\|^2 = \frac{1}{h^2}\left[(y_3 - y_1)^2 + (y_2 - y_3)^2\right]$$

Interpolation

Let's put some numbers considering this scenario



Interpolation

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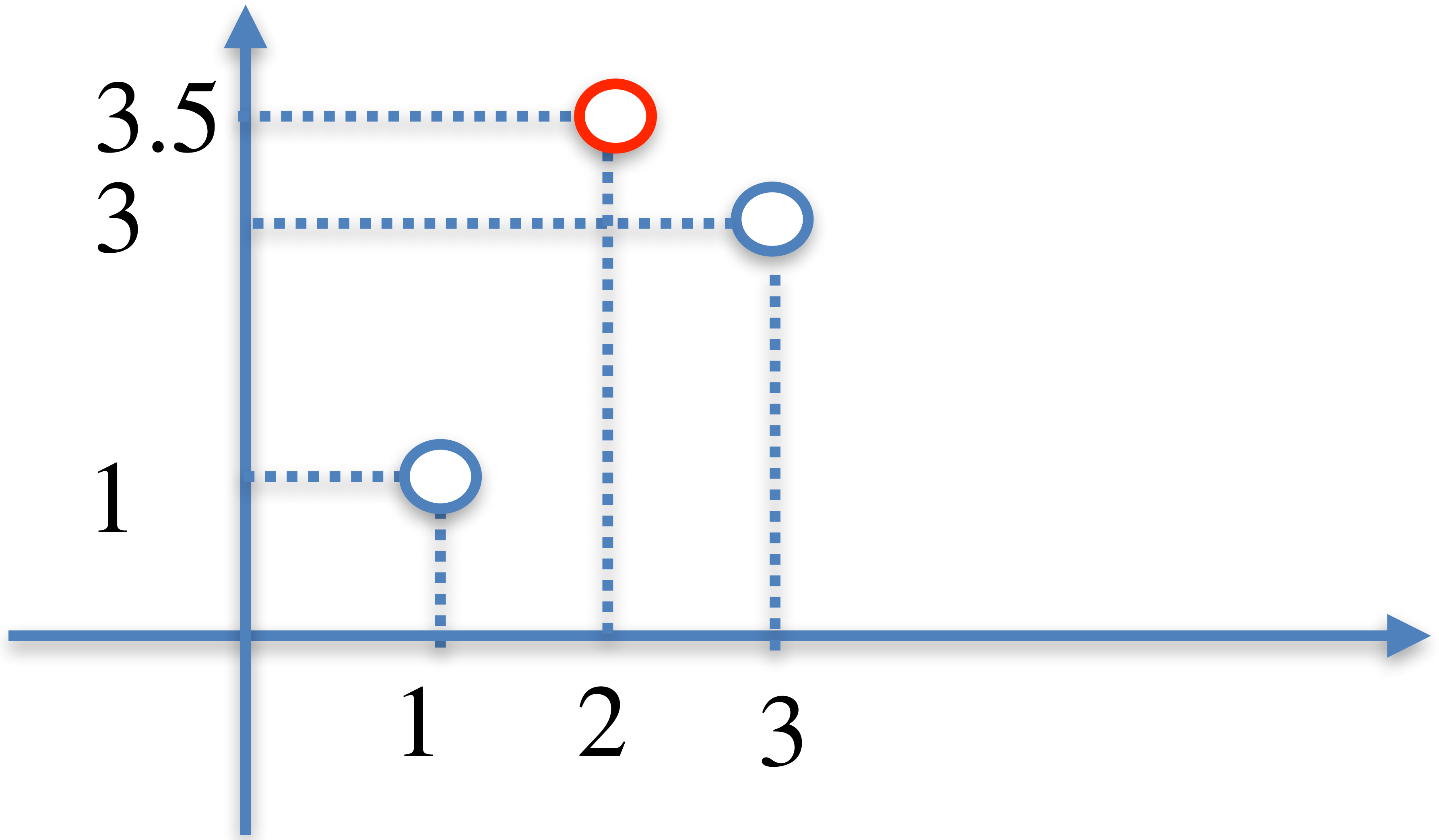

Interpolation

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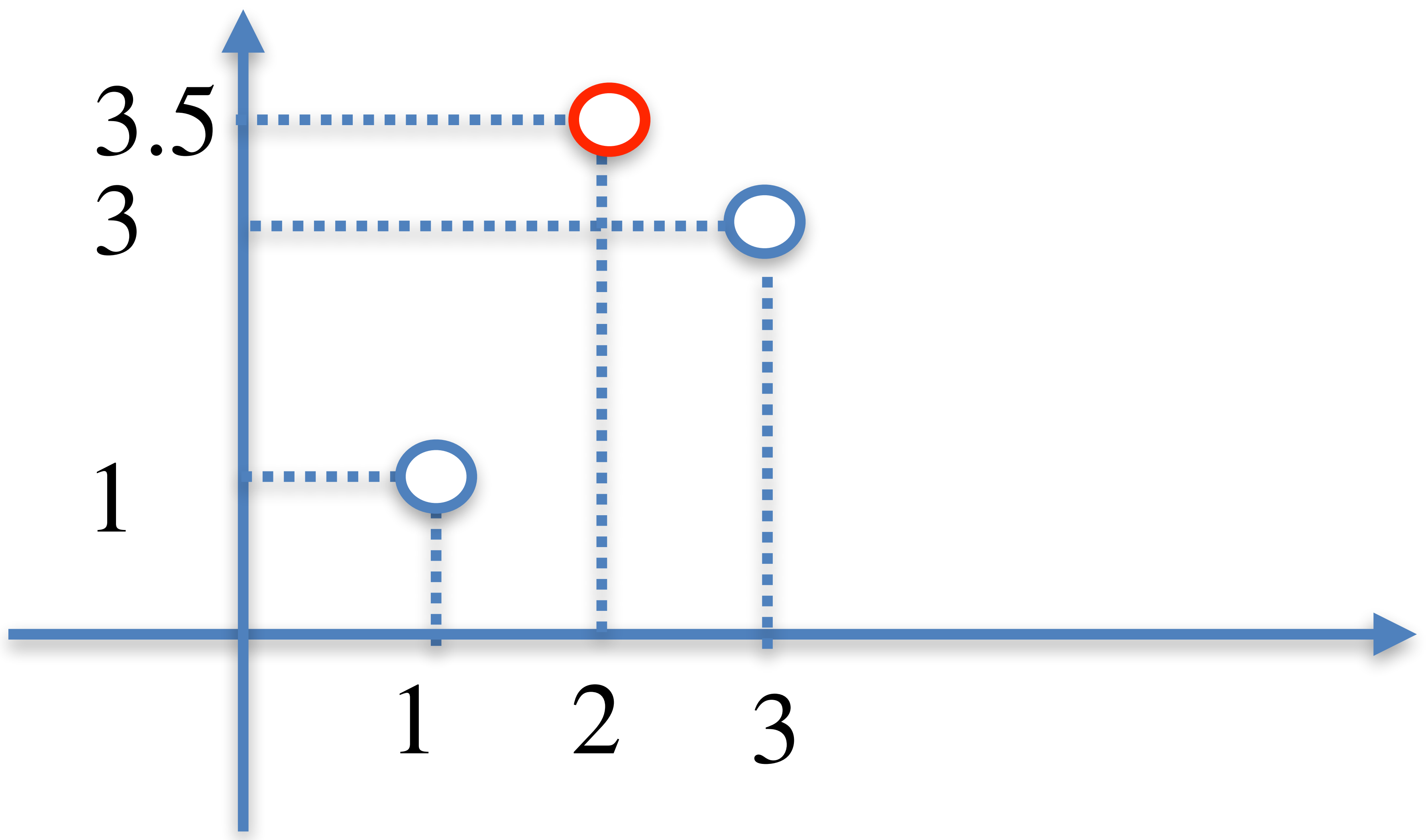
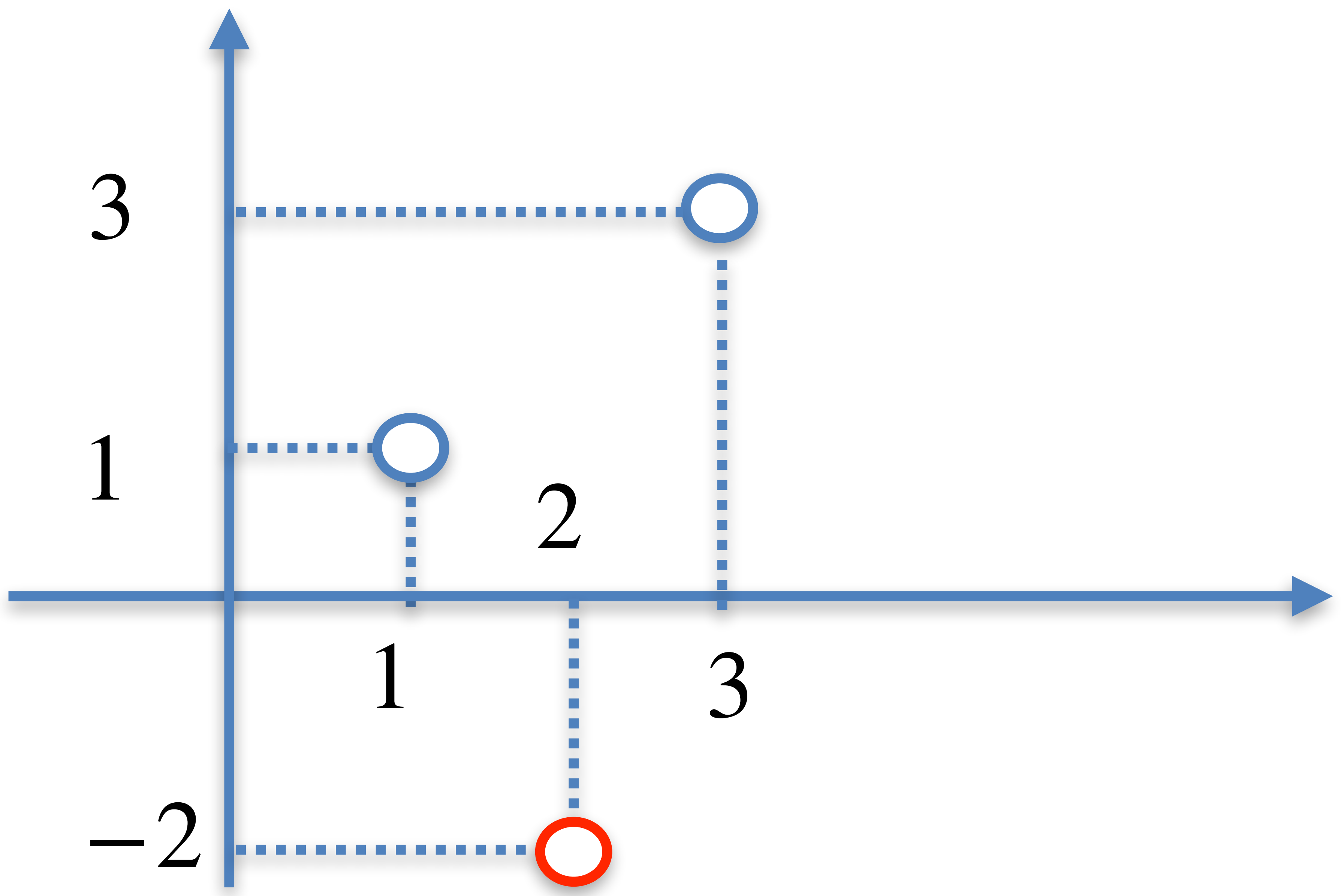
$$E(\mathbf{y}) = \left\| \mathbf{M}_{\frac{1}{h}} \mathbf{y} \right\|^2 \sim \frac{25}{4} + \frac{1}{4} = \frac{13}{2}$$



Interpolation

Let's put some numbers in this other scenario



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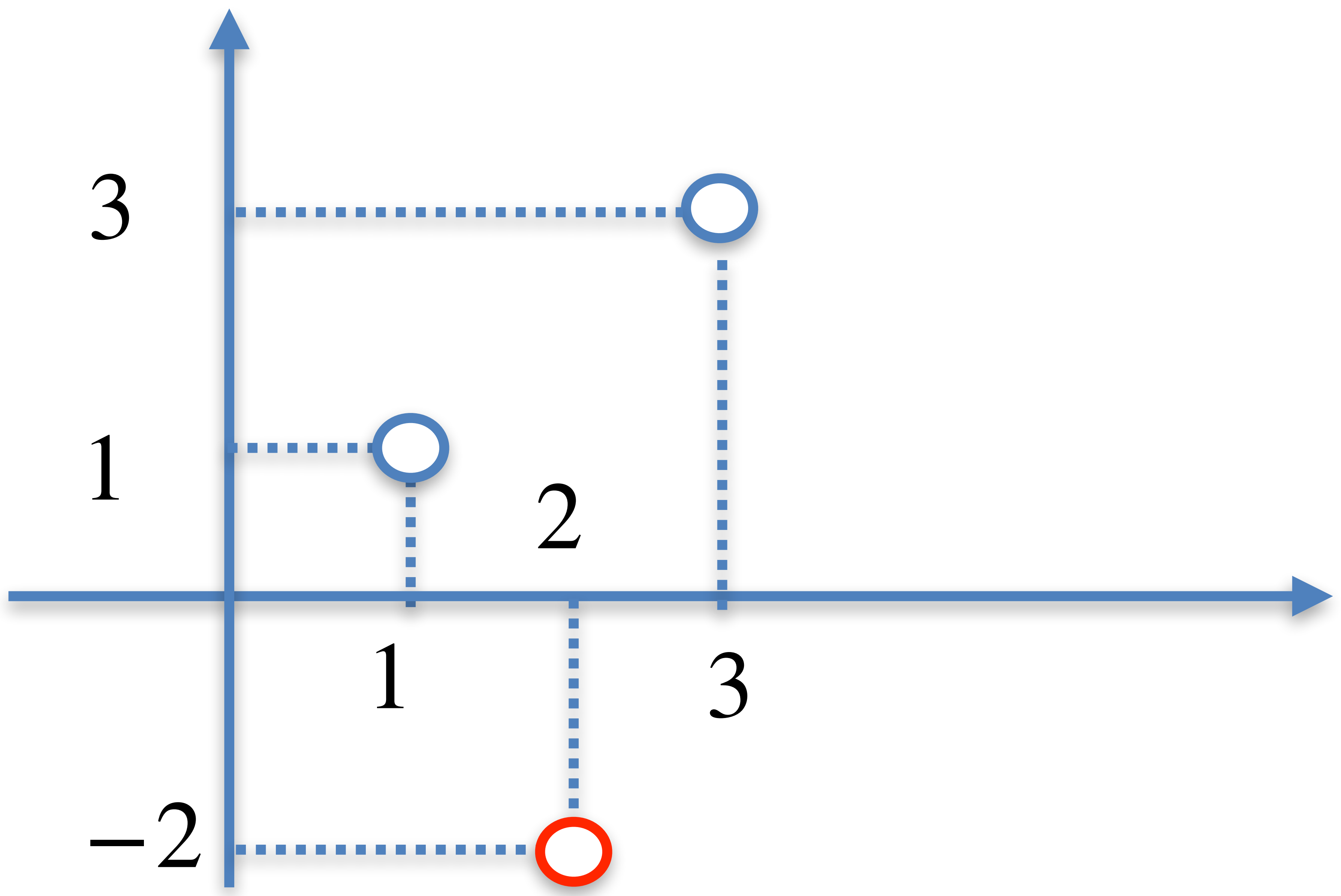

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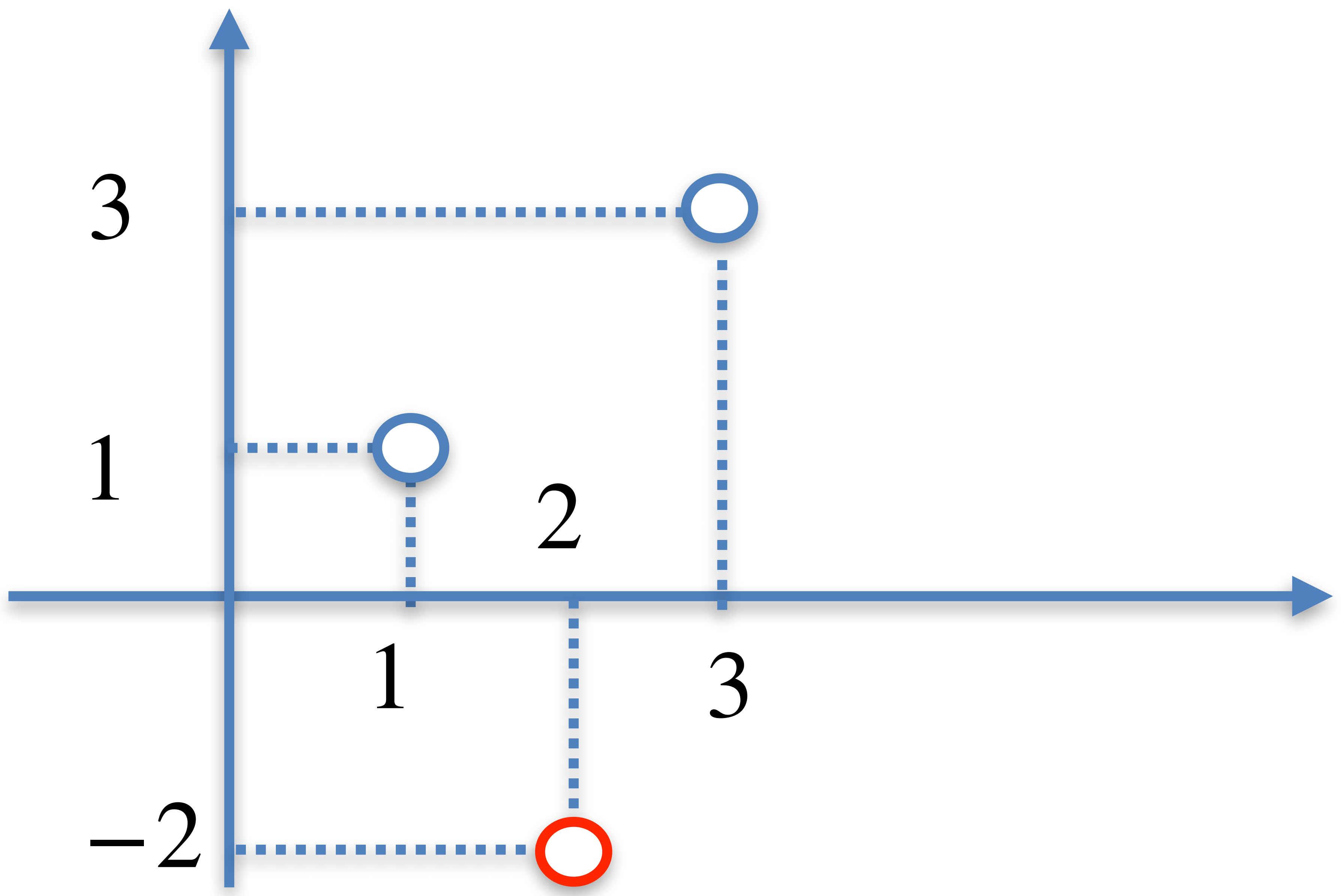
$$E(\mathbf{y}) = \left\| \mathbf{M}_{\frac{1}{h}}\, \mathbf{y} \right\|^2 \sim 9 + 25 = 34$$



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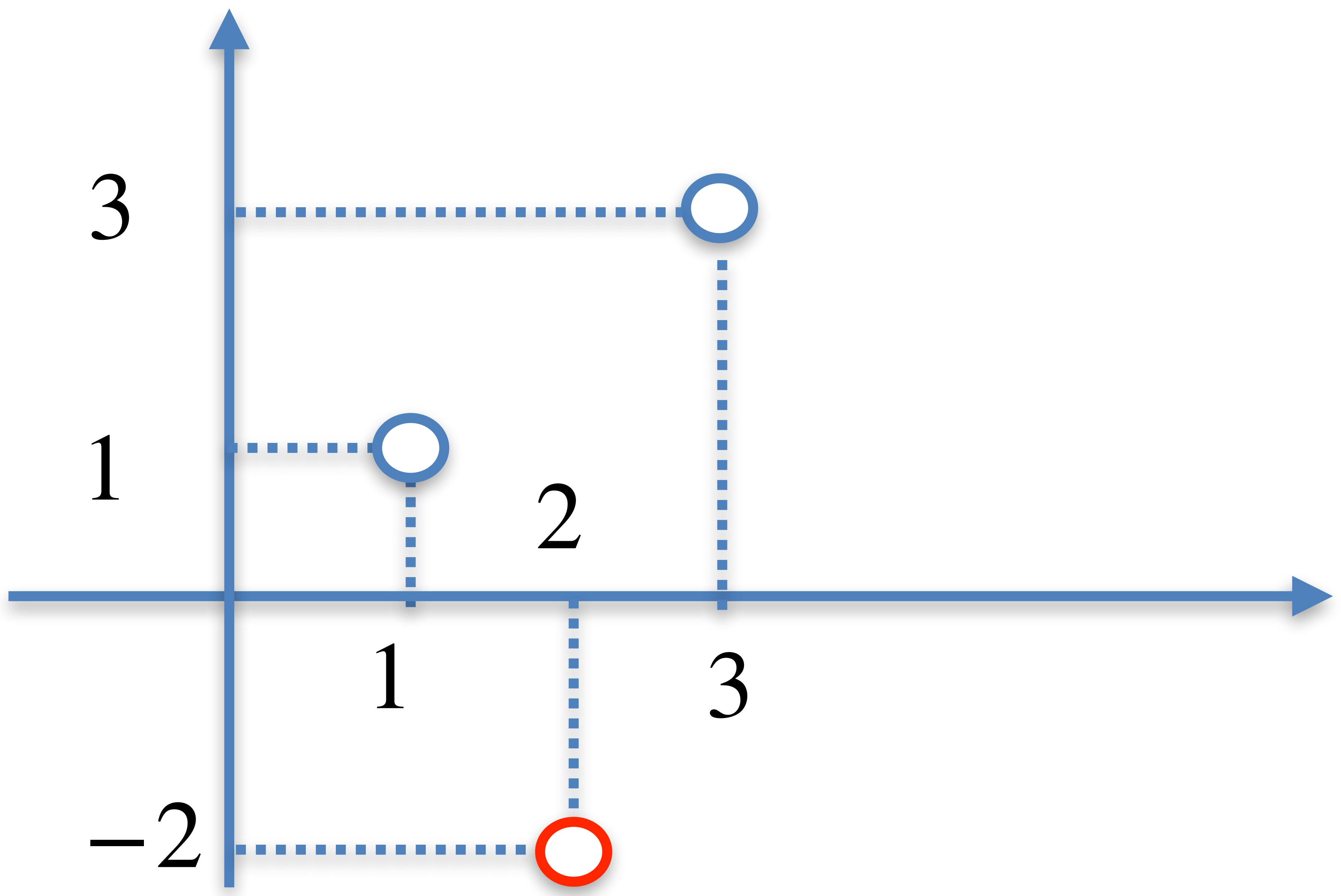

What is the minimum value?

Interpolation

The min can be found getting the derivative and setting to zero!

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$$\nabla E(\mathbf{y_3}) = 2(y_3 - y_1) - 2(y_3 - y_2) = 0$$

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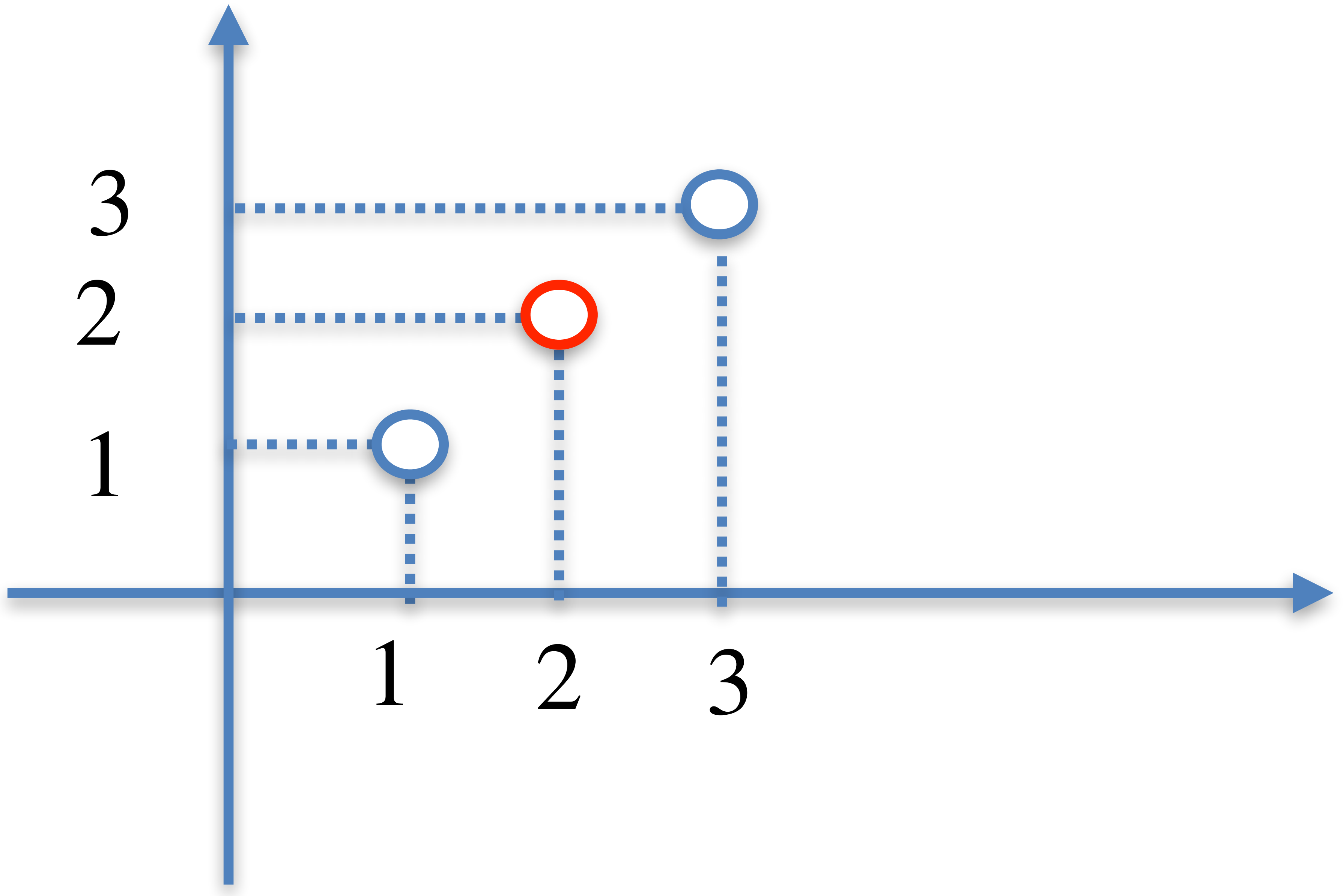

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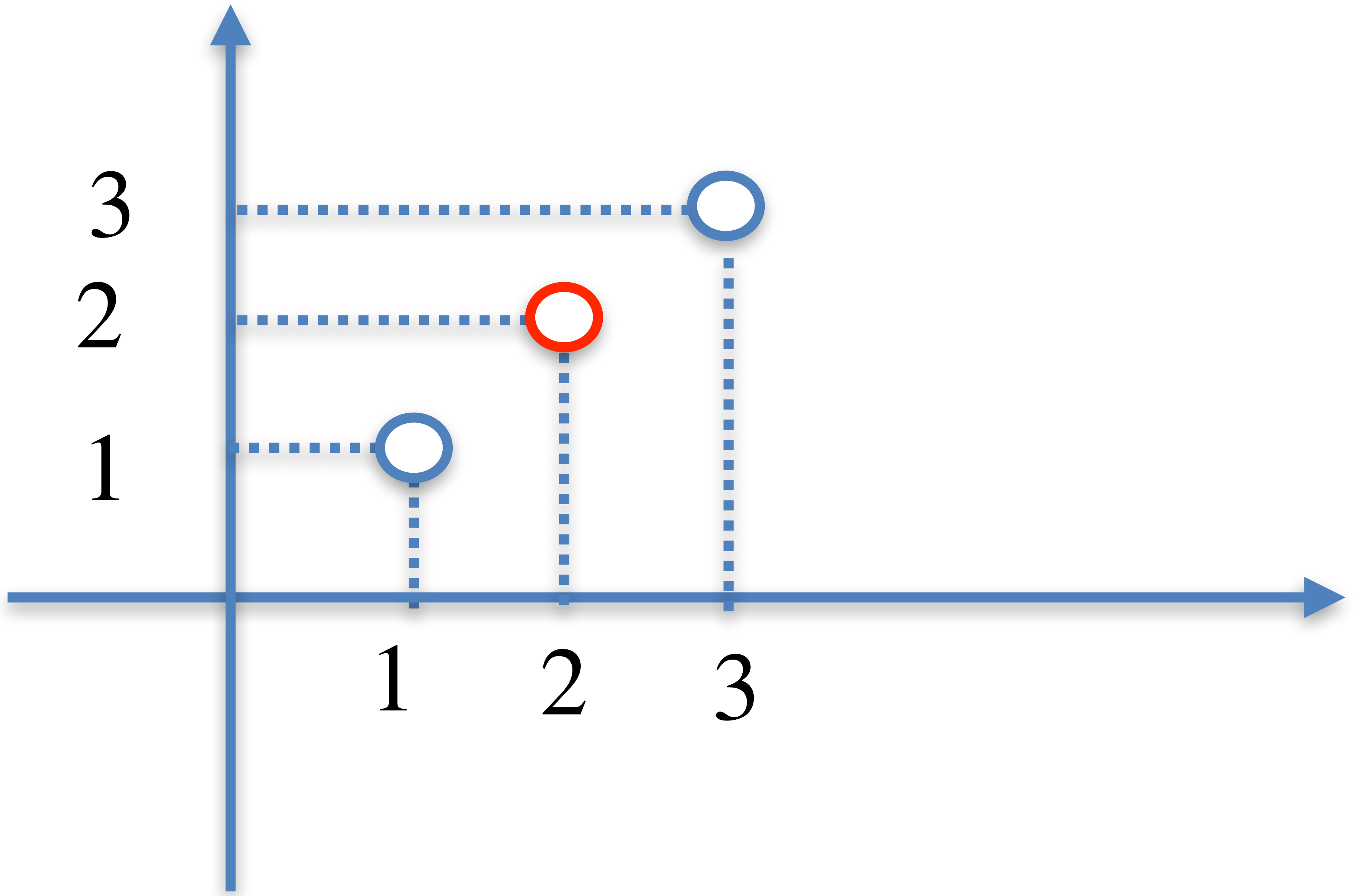

The min is, not surprisingly, the point laying in the middle between the two! This is why it is called interpolation!

Interpolation

So, it looks like that this energy function does the job!

$$E(\mathbf{y}) = \left\| \mathbf{M}_{\frac{1}{h}} \mathbf{y} \right\|^2 \quad \text{with} \quad \mathbf{M}_{\frac{1}{h}} = \frac{1}{h} \begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix},$$

Interpolation

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How can we use this to solve the problem in general?

Interpolation

We can write $\mathbf{y} = \mathbf{P}_{I_1 \backslash I_2} \mathbf{w} + \underbrace{\mathbf{P}_{I_2} \mathbf{v}}_{= \text{known}}$.

Interpolation

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Thus we split the vector $\mathbf{y}$ in two parts, the first of unknown the second of known

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The two P are projectors

Interpolation

Consider a simple case where we have only 2 known ys out of 5

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$$\begin{pmatrix} y_1 \\ y_2 \\ ? \\ ? \\ ? \end{pmatrix}$$

Interpolation

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$$\begin{pmatrix} y_1 \\ y_2 \\ ? \\ ? \\ ? \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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$$\mathbf{w} \in \mathbb{R}^3 \rightarrow \mathbf{P}_{I_1 \setminus I_2}\mathbf{w} \in \mathbb{R}^6$$

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$$\mathbf{v} \in \mathbb{R}^2 \rightarrow \mathbf{P}_{I_2} \mathbf{v} \in \mathbb{R}^6$$

Interpolation

Now, from $\mathbf{y} = \mathbf{P}_{I_1 \setminus I_2}\mathbf{w} + \underbrace{\mathbf{P}_{I_2}\mathbf{v}}_{= \text{known}}$.

Interpolation

Now, from $\mathbf{y} = \mathbf{P}_{I_1 \setminus I_2} \mathbf{w} + \underbrace{\mathbf{P}_{I_2} \mathbf{v}}_{= \text{known}}$.

The missing indices can be computed via

$$\min_{\{w_i\}_{i \in I_1 \setminus I_2}} \left\| \mathbf{M}_{\frac{1}{h}} \left(\mathbf{P}_{I_1 \setminus I_2} \mathbf{w} + \mathbf{P}_{I_2} \mathbf{v} \right) \right\|^2$$

Interpolation

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This is a least-squares problem, for which we know the solution. Indeed we can re-write

Interpolation

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Interpolation

$$\min_{\{w_i\}_{i\in I_1\setminus I_2}} \left\| \mathbf{M}_{\frac{1}{h}} \left(\mathbf{P}_{I_1\setminus I_2}\mathbf{w} + \mathbf{P}_{I_2}\mathbf{v} \right) \right\|^2$$

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Matrices are different but the form is the same of the usual MSE (except for the +!!)

Interpolation

The solution is then the normal equation with a minus on the right hand side

$$\min_{\{w_i\}_{i\in I_1\backslash I_2}} \left\| \mathbf{M}_{\frac{1}{h}} \left(\mathbf{P}_{I_1\backslash I_2}\mathbf{w} + \mathbf{P}_{I_2}\mathbf{v} \right) \right\|^2 = \min_{\{w_i\}_{i\in I_1\backslash I_2}} \left\| \mathbf{M}_{\frac{1}{h}}\mathbf{P}_{I_1\backslash I_2}\mathbf{w} + \mathbf{M}_{\frac{1}{h}}\mathbf{P}_{I_2}\mathbf{v} \right\|^2 = \min_{\{w_i\}_{i\in I_1\backslash I_2}} \left\| \mathbf{X}\mathbf{w} + \mathbf{r} \right\|^2$$

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Normal equation

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The solution is then the normal equation with a minus on the right hand side

$$\min_{\{w_i\}_{i \in I_1 \setminus I_2}} \left\| \mathbf{M}_{\frac{1}{h}} \left(\mathbf{P}_{I_1 \setminus I_2} \mathbf{w} + \mathbf{P}_{I_2} \mathbf{v} \right) \right\|^2 = \min_{\{w_i\}_{i \in I_1 \setminus I_2}} \left\| \mathbf{M}_{\frac{1}{h}} \mathbf{P}_{I_1 \setminus I_2} \mathbf{w} + \mathbf{M}_{\frac{1}{h}} \mathbf{P}_{I_2} \mathbf{v} \right\|^2 = \min_{\{w_i\}_{i \in I_1 \setminus I_2}} \left\| \mathbf{X}\mathbf{w} + \mathbf{r} \right\|^2$$

Normal equation $\mathbf{X}^\top \mathbf{X} \hat{\mathbf{w}} = -\mathbf{X}^\top \mathbf{r}$

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$$\mathbf{X}^\top\mathbf{X} = \mathbf{P}^\top_{I_1\backslash I_2}\mathbf{M}^\top_{\frac{1}{h}}\mathbf{M}_{\frac{1}{h}}\mathbf{P}_{I_1\backslash I_2} = \mathbf{P}^\top_{I_1\backslash I_2}\mathbf{L}_{\frac{1}{h}}\mathbf{P}_{I_1\backslash I_2}$$

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The solution is then the normal equation with a minus on the right hand side

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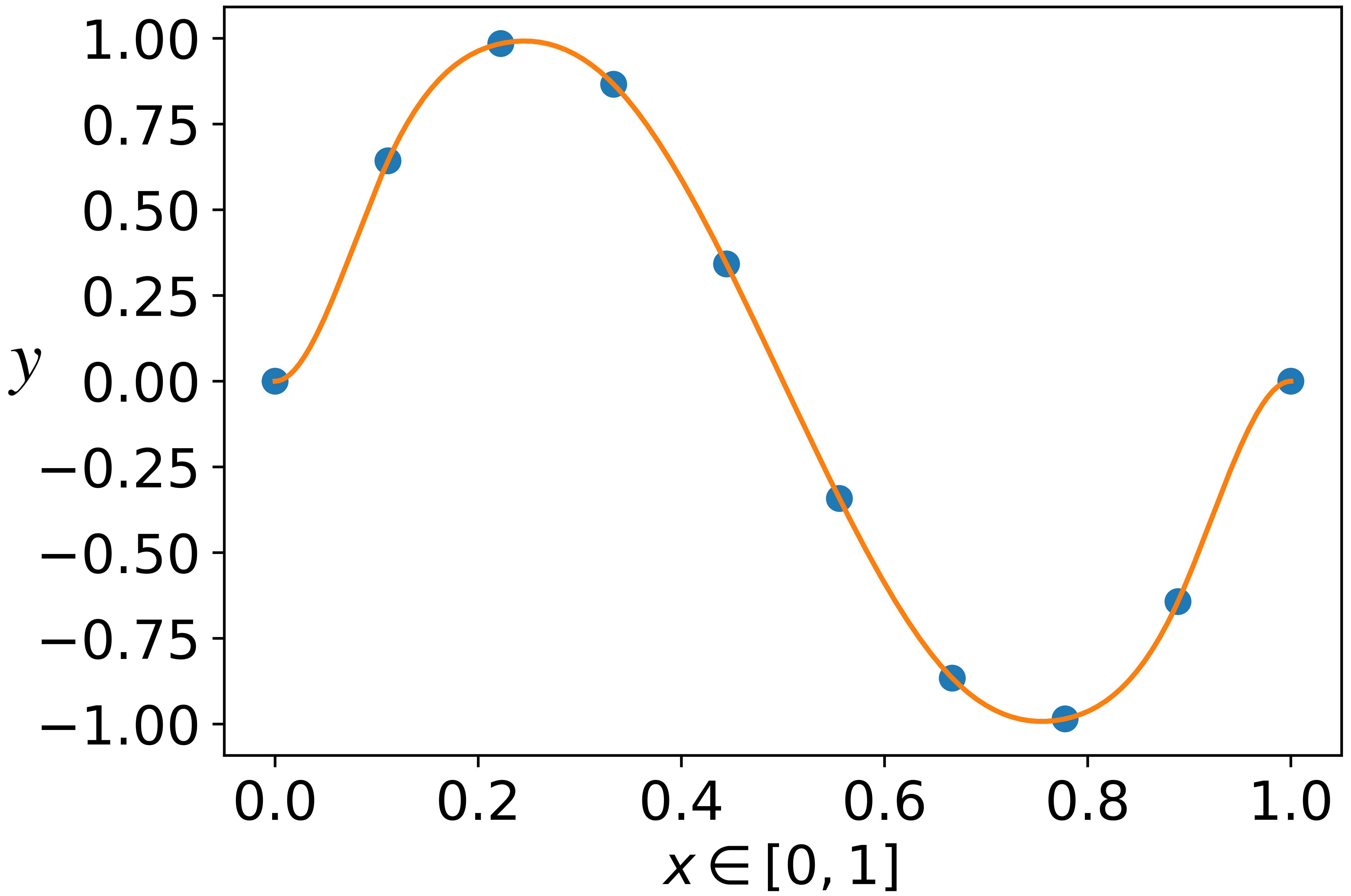
$$\mathbf{P}_{I_1\setminus I_2}^\top \mathbf{L}_{\frac{1}{h}} \mathbf{P}_{I_1\setminus I_2} \hat{\mathbf{w}} = \underbrace{-\mathbf{P}_{I_1\setminus I_2}^\top \mathbf{M}_{\frac{1}{h}}^\top \mathbf{M}_{\frac{1}{h}} \mathbf{P}_{I_2}\mathbf{v}}_{\text{known}} = -\mathbf{P}_{I_1\setminus I_2}^\top \mathbf{L}_{\frac{1}{h}} \mathbf{P}_{I_2}\mathbf{v},$$

Interpolation

Example

I_1 has 10000 points

I_2 has 10 points



"Training" set is very small, and since we don't know the ground truth for the others this a semi-supervised problem

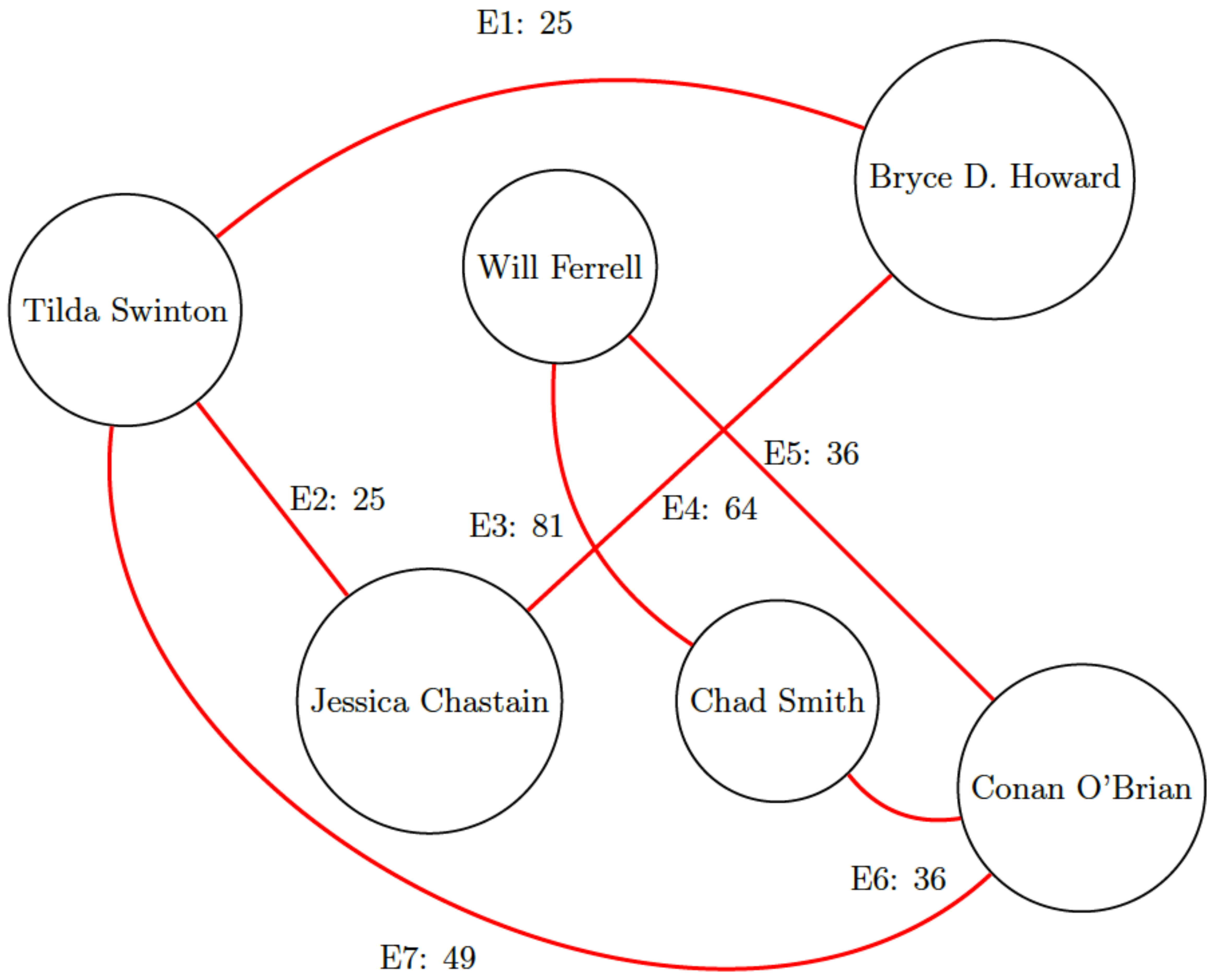
Applications

The advantages of using this formulation is that it can be applied to points like we just did, but also to data point for which you can define a similarity

Actors similarity

Incidence matrix

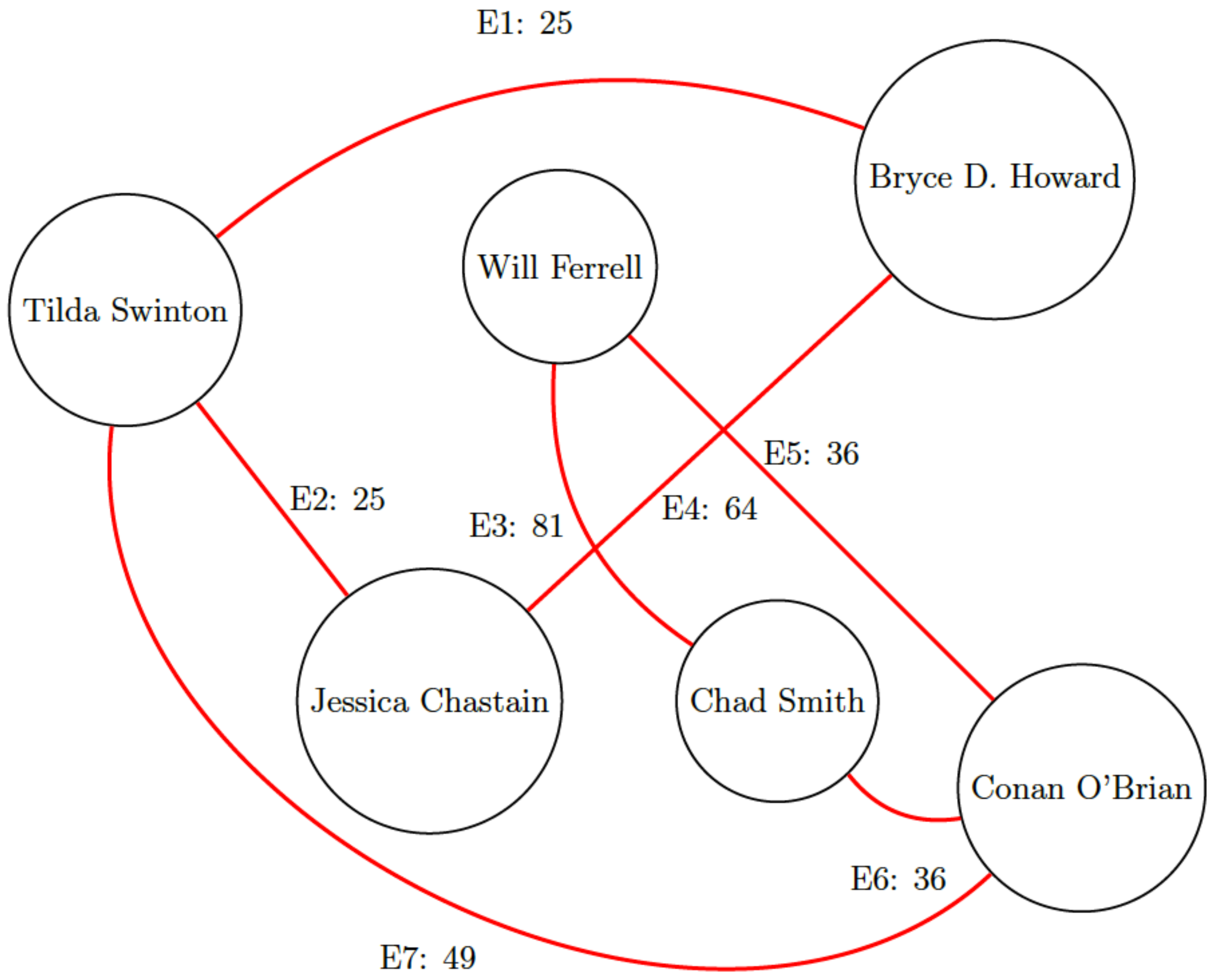
E1	−5	0	0	0	5	0
E2	0	0	0	−5	5	0
E3	0	−9	0	0	0	9
E4	−8	0	0	8	0	0
E5	0	0	−6	0	0	6
E6	0	−6	6	0	0	0
E7	0	0	−7	0	7	0
	B. D. Howard	C. Smith	C. O' Brian	J. Chastain	T. Swinton	W. Ferrell



Actors similarity

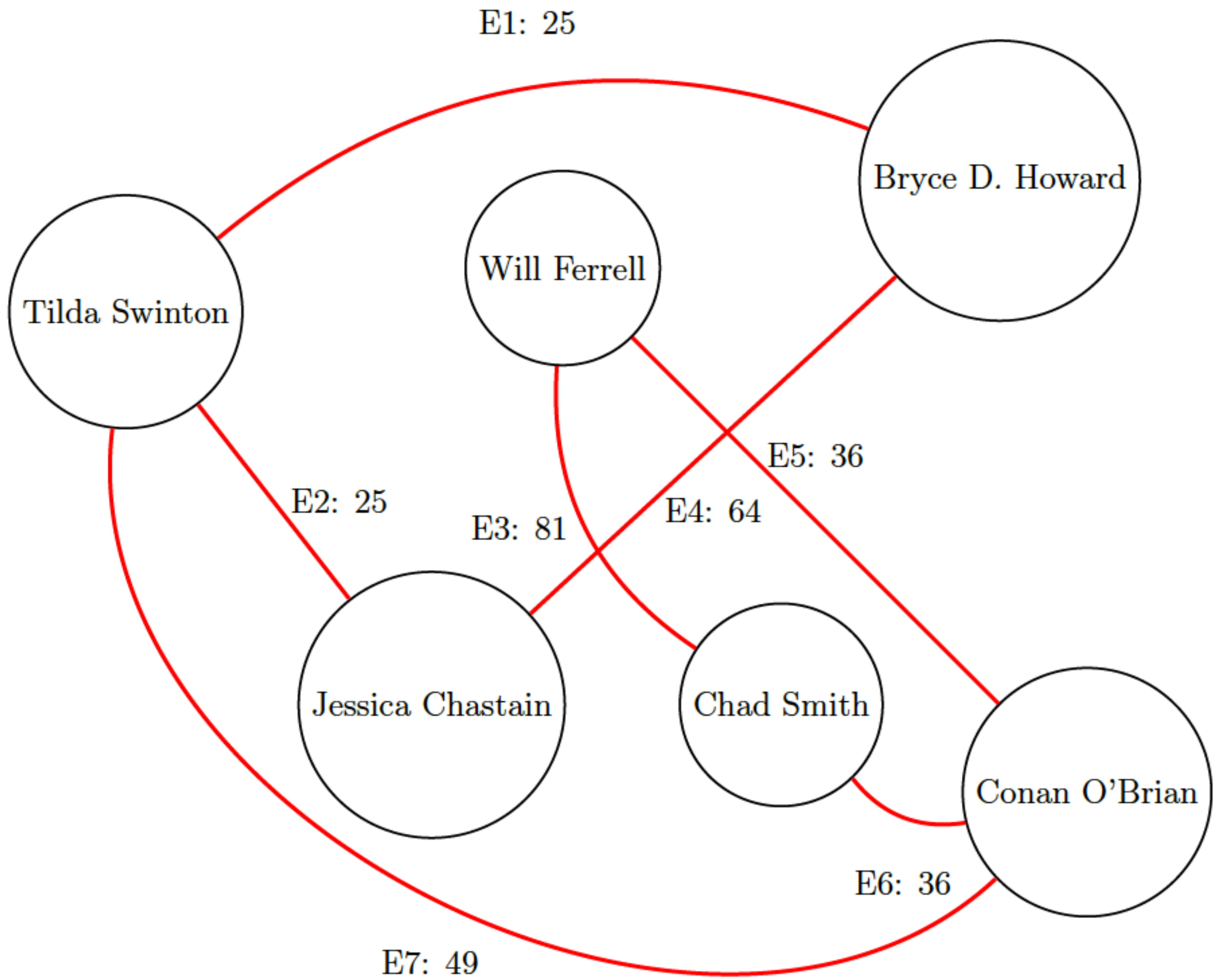
Laplacian matrix

$$L_w = M_w^\top M_w = \begin{pmatrix} 89 & 0 & 0 & -64 & -25 & 0 \\ 0 & 117 & -36 & 0 & 0 & -81 \\ 0 & -36 & 121 & 0 & -49 & -36 \\ -64 & 0 & 0 & 89 & -25 & 0 \\ -25 & 0 & -49 & -25 & 99 & 0 \\ 0 & -81 & -36 & 0 & 0 & 117 \end{pmatrix}.$$



Actors similarity

The task is: knowing the biological sex of a small set of actors, and using their similarity, predict the biological sex of the others



Actors similarity

Assume that we know that Jessica Chastain is female (label 1) and Will Ferrel male (label 0)

$$\mathbf{y}_{known} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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Actors similarity

Assume that we know that Jessica Chastain is female (label 1) and Will Ferrel male (label 0)

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2x1

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Assume that we know that Jessica Chastain is female (label 1) and Will Ferrel male (label 0)

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6x2 2x1

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$$\mathbf{y}_{known} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

6x6 6x2 2x1

$$\mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_1 \backslash I_2} \mathbf{y}_{unknown} = -\mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_2} \mathbf{y}_{known},$$

Actors similarity

Assume that we know that Jessica Chastain is female (label 1) and Will Ferrel male (label 0)

$$\mathbf{y}_{known} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

4x6 6x6 6x2 2x1

$$\mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_1 \backslash I_2} \mathbf{y}_{unknown} = - \mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_2} \mathbf{y}_{known} ,$$

Actors similarity

Assume that we know that Jessica Chastain is female (label 1) and Will Ferrel male (label 0)

$$\mathbf{y}_{known} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

4x1 4x6 6x6 6x2 2x1

$$\mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_1 \backslash I_2} \mathbf{y}_{unknown} = -\mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_2} \mathbf{y}_{known} ,$$

Actors similarity

Assume that we know that Jessica Chastain is female (label 1) and Will Ferrel male (label 0)

$$\mathbf{y}_{known} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

6x4 4x1 4x6 6x6 6x2 2x1

$$\mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_1 \backslash I_2} \mathbf{y}_{unknown} = -\mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_2} \mathbf{y}_{known},$$

Actors similarity

Assume that we know that Jessica Chastain is female (label 1) and Will Ferrel male (label 0)

$$\mathbf{y}_{known} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

6x6 6x4 4x1 4x6 6x6 6x2 2x1

$$\mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_1 \backslash I_2} \mathbf{y}_{unknown} = -\mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_2} \mathbf{y}_{known} \,,$$

Actors similarity

Assume that we know that Jessica Chastain is female (label 1) and Will Ferrel male (label 0)

$$\mathbf{y}_{known} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

4x6 6x6 6x4 4x1 4x6 6x6 6x2 2x1

$$\mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_1 \backslash I_2} \mathbf{y}_{unknown} = -\mathbf{P}_{I_1 \backslash I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_2} \mathbf{y}_{known},$$

Actors similarity

Actors similarity

$$\mathbf{P}_{I_2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Projects 2D vector in to 6D

Actors similarity

$$\mathbf{P}_{I_2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Projects 2D vector in to 6D

$$\mathbf{P}^{\top}_{I_1 \backslash I_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Projects 6D vector in to 4D selecting the unknown targets

Actors similarity

$$\mathbf{P}_{I_1 \setminus I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_1 \setminus I_2} \mathbf{y}_{unknown} = -\mathbf{P}_{I_1 \setminus I_2}^{\top} \mathbf{L}_w \mathbf{P}_{I_2} \mathbf{y}_{known} ,$$

$$\mathbf{P}_{I_2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Projects 2D vector in to 6D

$$\mathbf{P}_{I_1 \setminus I_2}^{\top} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Projects 6D vector in to 4D selecting the unknown targets

Actors similarity

Easy to find the solution

$$\begin{pmatrix} 89 & 0 & 0 & -25 \\ 0 & 117 & -36 & 0 \\ 0 & -36 & 121 & -49 \\ -25 & 0 & -49 & 99 \end{pmatrix} \tilde{v} = \begin{pmatrix} 64 \\ 0 \\ 0 \\ 25 \end{pmatrix} .$$

$$\hat{v} = \begin{pmatrix} 0.8912 & 0.0840 & 0.2732 & 1 & 0.6128 & 0 \end{pmatrix}^\top .$$

We can then impose a simple threshold >0.5 -> 1 <0.5 ->0