Queen Mary
University of London

MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 5 (Series)

1. Prove Lemma 4.6 from the lecture notes, i.e., show that if $\sum_{k=1}^{\infty} x_k = S$ and $c \in \mathbb{R}$ then $\sum_{k=1}^{\infty} cx_k = cS$.

Solution. We want to prove Lemma 4.6: If $\sum_{k=1}^{\infty} x_k = S$ and $c \in \mathbb{R}$ then $\sum_{k=1}^{\infty} cx_k = cS$.

Proof. If $\sum_{k=1}^{\infty} x_k = S$, this means that the partial sums $S_n = \sum_{k=1}^{n} x_k$ form a sequence $(S_n)_{n=1}^{\infty}$ which converges to S. But then by Theorem 3.24, the sequence $(cS_n)_{n=1}^{\infty}$ converges to cS. The claim therefore follows immediately by noting that $\sum_{k=1}^{n} cx_k = c\sum_{k=1}^{n} x_k$, which is simply the distributivity of $\mathbb{R}$ (note that we only have *finitely* many terms when applying the distributivity rule). $\square$

2. (a) Which of the following sums exist? Justify your answers, using any results from the lectures/notes.

$$\text{(i)}\ \sum_{k=1}^{\infty} \frac{1}{k^3}, \qquad \text{(ii)}\ \sum_{k=1}^{\infty} \frac{1}{k2^k}, \qquad \text{(iii)}\ \sum_{k=1}^{\infty} \frac{1}{3k}.$$

(b) Does the sum

$$\sum_{k=1}^{\infty} \left(\frac{1}{k^3} + \frac{1}{k2^k} - \frac{1}{3k} \right)$$

exist? Prove your assertion.

Solution.

(a) i. We know that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ exists. Also $0 \le \frac{1}{k^3} \le \frac{1}{k^2}$, so $\sum_{k=1}^{\infty} \frac{1}{k^3}$ exists by the comparison test.

ii. $0 \le \frac{1}{k2^k} \le \frac{1}{2^k} = \left(\frac{1}{2}\right)^k$. The sum $\sum_k (\frac{1}{2})^k$ exists, being the geometric series with common ratio $\frac{1}{2} < 1$ (see Theorem 4.11). By the comparison test, the sum $\sum_{k=1}^{\infty} \frac{1}{k2^k}$ exists.

iii. The sum $\sum_{k=1}^{\infty} \frac{1}{3k}$ does not exist. If it did, then so also would the sum $\sum_{k=1}^{\infty} \frac{1}{k}$, according to Lemma 4.6. But we know that the harmonic series does not converge.

(b) This sum does not exist.

Proof. Using (a) and Lemma 4.5, we know that the sum $\sum_k (\frac{1}{k^3} + \frac{1}{k2^k})$ exists. If the given sum would exist, then by Lemma 4.5 and 4.6, also the sum

$$\sum_{k=1}^{\infty} \frac{1}{3k} = \sum_{k=1}^{\infty} \left[\left(\frac{1}{k^3} + \frac{1}{k2^k} \right) - \left(\frac{1}{k^3} + \frac{1}{k2^k} - \frac{1}{3k} \right) \right]$$

would exist, but we have just seen in (b) that this is not the case. □

3. Use the ratio test to decide which of the following series exist:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + 3^k}{2^k + 5^k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + 5^k}{2^k + 3^k}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + 3^k + 5^k}{2^k + 3^k}$.

Solution.

(a) We write x_k for the individual terms in the series:

$$x_k = \frac{2^k + 3^k}{2^k + 5^k}$$

Note that these are all positive. We calculate:

$$\begin{aligned} \frac{x_{k+1}}{x_k} &= \frac{2^{k+1} + 3^{k+1}}{2^{k+1} + 5^{k+1}} \cdot \frac{2^k + 5^k}{2^k + 3^k} \\ &= \frac{2^{k+1} + 3^{k+1}}{2^k + 3^k} \cdot \frac{2^k + 5^k}{2^{k+1} + 5^{k+1}} \\ &= \frac{(\frac{2}{3})^{k+1} + 1}{\frac{1}{3}((\frac{2}{3})^k + 1)} \cdot \frac{\frac{1}{5}((\frac{2}{5})^k + 1)}{(\frac{2}{5})^{k+1} + 1} \\ &\to \frac{1}{1/3} \cdot \frac{1/5}{1} = \frac{3}{5} < 1. \end{aligned}$$

We conclude from the ratio test that the series converges. Parts (b) and (c) are similar. (Examples like this are very mechanical. You should do lots of them in order to gain fluency.)

4. Compute the value of the following series.

   (a) $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} = \sum_{k=1}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+2}\right)$.

   (b) $\sum_{k=1}^{\infty} \frac{2}{(k+10)(k+12)}$.

   Solution.

   (a) Write $S_n = \sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{1}{k+2}\right)$.
   Then we have $S_1 = \frac{1}{2} - \frac{1}{3}$, $S_2 = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = \frac{1}{2} - \frac{1}{4}$, etc. So

   $$S_n = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{14} - \frac{1}{5} + \ldots + \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{2} - \frac{1}{n+2}.$$

   Obviously (by Theorem 3.24), we have $S_n \to \frac{1}{2}$ as $n \to \infty$, so the series $\sum_{k=1}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+2}\right)$ exists and has value $\frac{1}{2}$.

   (b) Write $S_n = \sum_{k=1}^{n} \left(\frac{1}{k+10} - \frac{1}{k+12}\right)$.
   Then we have $S_1 = \frac{1}{11} - \frac{1}{13}$, $S_2 = \frac{1}{11} - \frac{1}{13} + \frac{1}{12} - \frac{1}{14}$, $S_3 = \frac{1}{11} - \frac{1}{13} + \frac{1}{12} - \frac{1}{14} + \frac{1}{13} - \frac{1}{15} = \frac{1}{11} + \frac{1}{12} - \frac{1}{14} - \frac{1}{15}$, etc. So this time, we have two positive and two negative terms that stay, while all others cancel each other. Hence, after these cancellations, we have

   $$S_n = \frac{1}{11} + \frac{1}{12} - \frac{1}{n+11} - \frac{1}{n+12}$$

   Obviously (by Theorem 3.24), we have $S_n \to \frac{1}{11} + \frac{1}{12} = \frac{23}{132}$ as $n \to \infty$, so the series $\sum_{k=1}^{\infty} \left(\frac{1}{k+10} - \frac{1}{k+12}\right)$ exists and has value $\frac{23}{132}$.

5. Prove that the series $\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$ converges for any $\alpha > 1$.

   [Hint. Try adapting the proof of Theorem 4.12, aiming this time for an *upper bound* on the partial sums.]

   Solution. As in the proof of Theorem 4.12, break up the partial sum S_{2^m} into blocks containing $1, 1, 2, 4, 8, \ldots, 2^{m-1}$ terms. The final block has 2^{m-1} terms, each bounded above by $1/(2^{m-1})^{\alpha}$. The sum of terms within the block is thus bounded above by $2^{m-1}(2^{m-1})^{-\alpha} = (2^{m-1})^{-(\alpha-1)} = 2^{-(m-1)(\alpha-1)}$. Thus $S_{2^m} = S_{2^{m-1}} + 2^{-(m-1)(\alpha-1)}$. Together with $S_1 = 1$, this gives

   $$\begin{aligned} S_{2^m} &\le 1 + 1 + 2^{-(\alpha-1)} + 2^{-2(\alpha-1)} + 2^{-3(\alpha-1)} + \cdots + 2^{-(m-1)(\alpha-1)} \\ &\le 1 + \frac{1}{1 - 2^{-(\alpha-1)}}. \end{aligned}$$

   where we have used the formula for the sum of a geometric series with common ratio $2^{-(\alpha-1)}$. (Note that we are using the fact hat α is strictly greater than 1.)

Thus (S_n) is an increasing sequence that is bounded above, which converges by Theorem 3.32. Thus $\sum_{k=1}^{\infty} \frac{1}{n^\alpha}$ exists.

In a sense, the harmonic series only "just converges".

6. Prove or disprove the following statements:

   (a) If $\sum_{k=1}^{\infty} x_k$ converges absolutely, then $\sum_{k=1}^{\infty} (-1)^k x_k$ exists.

   (b) If $\sum_{k=1}^{\infty} x_k$ converges absolutely, then $\sum_{k=1}^{\infty} \frac{x_k}{k}$ exists.

   (c) If $\sum_{k=1}^{\infty} x_k$ converges absolutely, then $\sum_{k=1}^{\infty} k \cdot x_k$ exists.

Solution. The parts (a) and (b) come from the Exam from May 2014.

(a) This is true.

Proof. If $\sum_k x_k$ converges absolutely, then $\sum_k |x_k|$ exists and thus $\sum_k |(-1)^k x_k| = \sum_k |x_k|$ exists. This means that $\sum_k (-1)^k x_k$ converges absolutely (by definition) and since absolute convergence implies convergence, the series exists. □

(b) This is true.

Proof. If $\sum_k x_k$ converges absolutely, then $\sum_k |x_k|$ exists and thus by the comparison test, using $0 \leq \frac{|x_k|}{k} \leq |x_k|$, the sum $\sum_k \frac{|x_k|}{k}$ exists. So $\sum_k \frac{x_k}{k}$ converges absolutely and hence $\sum_k \frac{x_k}{k}$ exists. □

(c) This is false.

Proof. A counterexample is $\sum_k x_k = \sum_k \frac{1}{k^2}$ which we know exists. As all terms are positive, the series also converges absolutely (for series with only positive terms, convergence and absolute convergence are obviously equivalent). However, $\sum_k k \cdot x_k = \sum_k \frac{1}{k}$, which we know does not exist. □

7. In this question, $\sum_{k=1}^{\infty} x_k$ is a series that converges absolutely.

   (a) Suppose that $|x_k| \leq 1$ for all $k \in \mathbb{N}$. Prove that the series $\sum_{k=1}^{\infty} x_k^2$ converges.

   (b) Now drop the assumption that $|x_k| \leq 1$ for all $k \in \mathbb{N}$. Prove that it is still the case that the series $\sum_{k=1}^{\infty} x_k^2$ converges.

Solution.

(a) Since $|x_k| \leq 1$, we have $0 \leq x_k^2 = |x_k|^2 \leq |x_k|$ for all $k \in \mathbb{N}$. We are given that $\sum_{k=1}^{\infty} |x_k|$ converges. So, by the comparison test, $\sum_{k=1}^{\infty} x_k^2$ converges.

(b) By Theorem 4.3, we know that the sequence $(|x_k|)_{k=1}^{\infty}$ converges to 0. This implies that the sequence $(|x_k|)_{k=1}^{\infty}$ is bounded, say, by $M > 0$. That is, $|x_k| \leq M$ for all $k \in \mathbb{N}$.

So we have $0 \leq x_k^2 = |x_k|^2 \leq M|x_k|$. We know from Lemma 4.6 that $\sum_{k=1}^{\infty} M|x_k|$ converges. So, by the comparison test, $\sum_{k=1}^{\infty} x_k^2$ converges.

8. What happens in the previous question if we drop the word 'absolutely', so that $\sum_{k=1}^{\infty} x_k$ is a series that merely converges?

Solution. Let $x_k = (-1)^k/\sqrt{k}$ and consider the series

$$\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} (-1)^k/\sqrt{k}.$$

This series converges using the same argument that was used for the alternating harmonic series. (Check this!) However, $\sum_{k=1}^{\infty} x_k^2 = \sum_{k=1}^{\infty} 1/k$, which is the non-convergent harmonic series. Note that the series $\sum_{k=1}^{\infty} (-1)^k/\sqrt{k}$ is not absolutely convergent.

9. Let $x_k = \frac{2}{k(k+1)(k+2)}$ for all $k \in \mathbb{N}$ and define $S_n = \sum_{k=1}^{n} x_k$.

(a) Evaluate S_n as a function of n.

Hint. In a similar situation, in the notes and lectures, we used the fact that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Try something similar here, writing $\frac{2}{k(k+1)(k+2)}$ as a difference of two simpler quotients.

(b) Evaluate the limit of the sequence $(S_n)_{n=1}^{\infty}$.

Note that the limit from part (b) is by definition $\sum_{n=1}^{\infty} \frac{2}{k(k+1)(k+2)}$.

Solution.

(a) We use the fact that

$$\frac{2}{k(k+1)(k+2)} = \frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}.$$

Then

$$\begin{aligned}S_n &= \Big(\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3}\Big) + \Big(\frac{1}{2\cdot 3} - \frac{1}{3\cdot 4}\Big) + \\ &\qquad \cdots + \Big(\frac{1}{(n-1)n} - \frac{1}{n(n+1)}\Big) + \Big(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\Big) \\ &= \frac{1}{1\cdot 2} + \Big(-\frac{1}{2\cdot 3} + \frac{1}{2\cdot 3}\Big) + \cdots + \Big(-\frac{1}{n(n+1)} + \frac{1}{n(n+1)}\Big) - \frac{1}{(n+1)(n+2)} \\ &= \frac{1}{2} - \frac{1}{(n+1)(n+2)}.\end{aligned}$$

(b) Since $\left|\frac{1}{(n+1)(n+2)}\right| \le \left|\frac{1}{n}\right|$, the sequence $\left(\frac{1}{(n+1)(n+2)}\right)$ converges to 0, by dominated convergence. Thus, (S_n) converges to $\frac{1}{2}$ by Theorem 3.24.

10. Here we study $\sum_{k=1}^{\infty} \frac{2^k}{k!}$.

(a) Show that $k! \ge 3^{k-2}$.

(b) Deduce that $2^k/k! \le 4 \cdot (2/3)^{k-2}$.

(c) Deduce that $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ exists.

Solution. Here we look at $\sum_{k=1}^{\infty} 2^k/k!$.

(a) $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot k \ge 1 \cdot 1 \cdot 3 \cdot 3 \cdot \ldots \cdot 3 = 3^{k-2}$.

(b) Using a), we immediately deduce $2^k/k! \le 2^k/3^{k-2} = 4 \cdot (2/3)^{k-2}$.

(c) We have $0 \le 2^k/k! \le 4 \cdot (2/3)^{k-2}$ (from (b)) and the geometric series $\sum_{k=1}^{\infty} 4 \cdot (2/3)^{k-2} = 4 \sum_{k=1}^{\infty} (2/3)^{k-2}$ exists (you should remember this from Calculus, but we will also prove this in the next few days in the lectures). So, the partial sums of the series $\sum_{k=1}^{\infty} 2^k/k!$ form an increasing and bounded sequence and hence converge.

[There is also an easier argument using the comparison test. Try to find it.]

11. Here we study $\sum_{k=1}^{\infty} \frac{k!}{k^k}$.

(a) Prove that $k! \le k^k$.

(b) Prove that $k! \le 2k^{k-2}$.

(c) Deduce from (b) that $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ exists.

Solution. Here we look at $\sum_{k=1}^{\infty} k!/k^k$.

(a) $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot k \leq k \cdot k \cdot k \cdot k \cdot \ldots \cdot k = k^k$.

(b) We can improve the estimate from a) slightly by keeping the first two elements: $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot k \leq 1 \cdot 2 \cdot k \cdot k \cdot \ldots \cdot k = 2k^{k-2}$.

(c) We have $0 \leq k!/k^k \leq 2k^{k-2}/k^k = 2/k^2$ and one can show that $\sum_{k=1}^{\infty} 2/k^2$ exists. (We will do this in the next few days in the lectures.) So, again, the partial sums of $\sum_{k=1}^{\infty} k!/k^k$ form an increasing and bounded sequence and hence converge.

[Again, the comparison test would give an easier proof].

Note: In question 10(c) and 11(c), you were confronted with material which we haven't yet studied in this lecture course yet, but you should remember these results from your Calculus class. I have included these questions on purpose to train you to "think outside of the box" and, if necessary, look things up elsewhere – important qualities of any independent mathematician and important employability skills. (Of course, I am only doing this because the homework is not contributing to your final mark.) Of course, you could also *prove* that the geometric series in 6(c) and the series $\sum_{k=1}^{\infty} 2/k^2$ in 7(c) does indeed exist directly from the definition. In this case however, these questions would be rather hard... but of course we will do exactly that in the lecture course soon!

12. Assume that $\sum_{k=1}^{\infty} x_k$ converges and that $(y_k)_{k=1}^{\infty}$ is a bounded sequence.

(a) Find a counterexample to the statement: "$\sum_{k=1}^{\infty} x_k y_k$ converges."

(b) Prove that if we additionally assume that $x_k \geq 0$ for all $k \in \mathbb{N}$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

Solution.

(a) There are many counterexamples to the statement $\sum_{k=1}^{\infty} x_k y_k$ converges. The easiest is when $\sum_{k=1}^{\infty} x_k$ converges, but does not converge absolutely, and $(y_k)_{k=1}^{\infty}$ is a sequence consisting of the values 1 and -1 in such a way that $x_k y_k = |x_k|$.

A concrete example is given by the following: Let $x_k = (-1)^{k+1}\frac{1}{k}$. Then (as we have seen in class), $\sum_{k=1}^{\infty} x_k$ is the alternating harmonic series, which we know exists (and has value $\log 2$). Moreover, let $y_k = (-1)^{k+1}$. Clearly this is bounded by -1 and 1. Moreover, $x_k y_k = (-1)^{2k+2}\frac{1}{k} = \frac{1}{k}$, so $\sum_{k=1}^{\infty} x_k y_k = \sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series, which we have seen does not converge.

(b) If $x_k \geq 0$, then $|x_k| = x_k$, so $\sum_{k=1}^{\infty} x_k$ converges absolutely. As (y_k) is a bounded sequence, there exists some $M \in \mathbb{R}$ such that $|y_k| \leq M$ for all $k \in \mathbb{N}$. Now for this M, we get from Lemma 4.6 that $\sum_{k=1}^{\infty} M \cdot x_k$ converges absolutely. Moreover, as $0 \leq |x_k y_k| \leq M|x_k|$, the comparison test shows that $\sum_{k=1}^{\infty} |x_k y_k|$ converges, and because absolute convergence implies convergence, we have that $\sum_{k=1}^{\infty} x_k y_k$ converges. This proves the claim.

13. For which values of $x \in \mathbb{R}$ do the following power series exist? Give a precise answer and justify it.

$$\text{(a)} \quad \sum_{k=1}^{\infty} 2^k x^k \qquad \text{and} \qquad \text{(b)} \quad \sum_{k=1}^{\infty} \frac{2^k x^k}{k}.$$

Solution.

(a) Writing $\sum_{k=1}^{\infty} 2^k x^k$ as $\sum_{k=1}^{\infty} (2x)^k$ we see it as a geometric series with common ratio $2x$. So the series converges when $|2x| < 1$, i.e., $|x| < \frac{1}{2}$, and does not converge when $|2x| \geq 1$, i.e., $|x| \geq \frac{1}{2}$,

(b) When $x = \frac{1}{2}$, the series becomes the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ which we know does not converge. Thus the radius of convergence R of the sequence satisfies $R \leq \frac{1}{2}$ (Theorem 4.30). When $x = -\frac{1}{2}$, the series becomes the (negation of the) alternating harmonic series $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ which we know does converge. Thus the radius of convergence satisfies $R \geq \frac{1}{2}$ (Theorem 4.30 again). Summarising, the series converges when $|x| < \frac{1}{2}$ and when $x = -\frac{1}{2}$, and does not converge when $|x| > \frac{1}{2}$ and when $x = \frac{1}{2}$.

14. For which values of $x \in \mathbb{R}$ do the following power series exist? Give a precise answer and justify it.

$$\text{(a)} \quad \sum_{k=1}^{\infty} \frac{x^k}{k^2} \qquad \text{and} \qquad \text{(b)} \quad \sum_{k=1}^{\infty} k x^k.$$

exist?

[Hint. One possibility for (b) is to use the easily checked inequality $k \leq \alpha^{-1}(1+\alpha)^k$, valid for all $\alpha > 0$ and $k \in \mathbb{N}$.]

Solution.

(a) This power series converges for all x with $-1 \leq x \leq 1$.

Again, one can argue directly: For $|x| \leq 1$, we have $0 \leq \frac{|x|^k}{k^2} \leq \frac{1}{k^2}$ and thus convergence follows from the comparison test. If $|x| > 1$, then $(\frac{x^k}{k^2})$

is a sequence which does not converge to zero and hence the series cannot exist, by Theorem 4.3.

To expand on the last step: Let $z_k = \frac{x^k}{k^2}$, so that $(z_k)_{k=1}^{\infty}$ is the sequence of interest. Consider the sequence of ratios (z_k/z_{k+1}). Note that

$$\frac{z_k}{z_{k+1}} = \frac{1}{x}\frac{(k+1)^2}{k^2} = \frac{1}{x}\Big(1 + \frac{2}{k} + \frac{1}{k^2}\Big).$$

Using by now familiar arguments, $z_k/z_{k+1} \to 1/x$ and hence $z_{k+1}/z_k \to x$. If $x > 1$ then there exists $K \in \mathbb{N}$ such that $z_{k+1}/z_k > 1$ for all $k > K$. So the sequence (z_k) is increasing for $k > K$ and does not converge to 0 (and indeed tends to ∞).

(b) When $|x| \geq 1$, we have that the sequence $(kx^k)_{k=1}^{\infty}$ tends to ∞, and in particular, the sequence does not converge to 0. Therefore the series $\sum_{k=1}^{\infty} kx^k$ does not converge in this instance.

If $|x| < 1$, choose $\alpha > 0$ to satisfy $(1+\alpha)|x| < 1$; e.g., let $\alpha = (1-x)/2x$. Now observe that

$$0 < |kx^k| = k|x|^k \leq \alpha^{-1}(1+\alpha)^k|x|^k = \alpha^{-1}\big((1+\alpha)|x|\big)^k.$$

The series $\sum_{k=1}^{\infty}\big((1+\alpha)|x|\big)^k$, being a geometric series with common ratio less than 1, converges. By Lemma 4.6 and the comparison test, $\sum_{k=1}^{\infty} kx^k$ converges.