Lecture 2

We write the summary here.

- Pell's equation $x^2 - dy^2 = \pm 1$
- Let $\sqrt{d} = [a; \overline{a_1, a_2, \ldots, a_\ell}]$
- Thm 2: $(s_{n\ell-1}, t_{n\ell-1})$ are solutions to $x^2 - dy^2 = (-1)^{n\ell}$
- Thm 3 + Thm 4: $v_n + \sqrt{d}\, w_n = (s + t\sqrt{d})^n$ where (s, t) is the fundamental solution.

$$\{ \text{Sol}^n\text{s of } x^2 - dy^2 = \pm 1 \} = \{ (v_n, w_n) \}$$

In particular, 1) $(s_{\ell-1}, t_{\ell-1}) = (v_1, w_1)$ is the fundamental solution

2) $(v_n, w_n) = (s_{n\ell-1}, t_{n\ell-1})$

$\Rightarrow v_n^2 - d w_n^2 = (-1)^{n\ell}$.

Thus if ℓ is odd then (v_n, w_n) are solutions to $x^2 - dy^2 = -1$ for odd n.

(v_n, w_n) are solutions to $x^2 - dy^2 = +1$ for even n.

- If l is even then there is no solution to $x^2 - dy^2 = -1$, (v_n, w_n) for any n are solutions to $x^2 - dy^2 = +1$.

<u>Ex</u>: Find all solutions to $x^2 - 17y^2 = \pm 1$.

<u>Ans</u>: $\lfloor \sqrt{17} \rfloor = 4$

$$\frac{1}{\sqrt{17} - 4} = \sqrt{17} + 4, \quad \lfloor \sqrt{17} + 4 \rfloor = 8$$

So $\sqrt{17} = [4; \overline{8}] = [4; 8, 8, \ldots]$

$s_{-2} = 0$, $s_{-1} = 1$, $s_0 = 4$, $s_1 = 8 \times 4 + 1 = 33$

$t_{-2} = 1$, $t_{-1} = 0$, $t_0 = 1$, $t_1 = 8 \times 1 + 0 = 8$

Indeed, $(4, 1)$ is fundamental solution of $x^2 - 17y^2 = \pm 1$ (in fact -1)

In fact, $33^2 - 17 \times 8^2 = 1$

$(33, 8)$ is minimal solution for

$x^2 - 17y^2 = 1$.

we can check $(4 + \sqrt{17})^2$

$= 16 + 17 + 8\sqrt{17} = 33 + 8\sqrt{17}$.

<u>Exercise</u>: Find minimal positive solution to $x^2 - 10y^2 = 1$.

<u>Exercise</u>: Find all solutions to

$x^2 - 11y^2 = \pm 1$

<u>Exercise</u>: If $s + t\sqrt{d} < s' + t'\sqrt{d}$

and $s^2 - dt^2 = s'^2 - dt'^2 = \pm 1$

$s - \sqrt{d}\, t > s' - t'\sqrt{d}$

$\Rightarrow \quad 2t\sqrt{d} < 2t'\sqrt{d} \Rightarrow t < t'$

$\Rightarrow \quad s^2 - s'^2 = d(t^2 - t'^2) < 0$

$\Rightarrow \quad s < s'$ as $s, s' > 0$