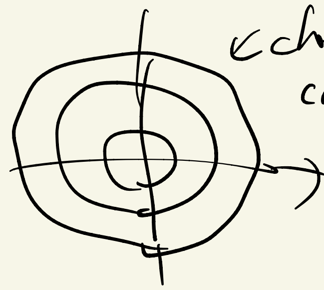
PS2 Q9:

The characteristics are $\frac{dt}{dx} = \frac{x}{-t}$

i.e. $-t\,dt = x\,dx$

$$-\frac{t^2}{2} = \frac{x^2}{2} + \tilde{c}$$

$$t^2 + x^2 = c$$



Along the characteristics, the PDE becomes an ODE

$$\frac{d}{dx} u(x, t(x)) = u_x - \frac{dt}{dx} u_t$$

$$= u_x - \frac{x}{-t} u_t$$

$$= \frac{t u_x - x u_t}{t}$$

$$= \frac{-1}{t}\left[x u_t - t u_x\right]$$

$$= -\frac{1}{t}$$

↑ using the PDE

using $x^2 + t^2 = c$

we get $t = \sqrt{c - x^2}$

So the ODE is

$$\frac{du}{dx} = \frac{-1}{\sqrt{c - x^2}}$$

Solve it we get

$$u = -\arcsin\frac{x}{\sqrt{C}} + f(C)$$

using that $C = t^2+x^2$ for the characteristic equations. we get

$$u(x,t) = -\arcsin\frac{x}{\sqrt{t^2+x^2}} + f(x^2+t^2)$$

$$= -\arctan\left(\frac{x}{t}\right) + f(x^2+t^2)$$

using the boundary conditions, we have

$$0 = u(0,t) = -\arctan 0 + f(t^2)$$

we get $f(t^2) = 0$

replacing t by $\sqrt{t}$, we get

$$f(t) \equiv 0$$

So the solution to the BVP is

$$u(x,t) = -\arctan\left(\frac{x}{t}\right).$$