WEEK 10 NOTES

We continue studying heat equations this week.

1. Heat equations on an interval (continued)

For the heat equation on an interval with Dirichlet boundary conditions

$$\begin{aligned}
&U_t = \varkappa U_{xx}, \qquad x \in [0, L], \quad t > 0, \\
&U(x, 0) = f(x), \\
&U(0, t) = 0, \qquad U(L, t) = 0.
\end{aligned}$$

We showed last week using separation of variables that the general solutions are

$$U(x, t) = \sum_{n=1}^{\infty} a_n e^{-\frac{\pi^2 n^2}{L^2}\varkappa t} \sin\left(\frac{\pi n x}{L}\right). \tag{1.1}$$

We will next use the initial condition $U(x, 0) = f(x)$ to determine the a_n coefficients.

1.1. **Initial conditions.**

The condition $U(x, 0) = f(x)$ with $0 < x < L$ fixes the solution. Evaluating (1.1) at $t = 0$ one has

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n x}{L}\right).$$

This is a Fourier sine series —we have already found these series a couple of times before. The coefficients a_n are then determined via the Fourier coefficients —thus,

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx.$$

1.2. **Examples.**

We now look at some concrete examples of the discussion in the previous paragraphs.

Example 1.1. Let the initial conditions be given by

$$f(x) = \sin\left(\frac{\pi x}{L}\right).$$

It follows then that

$$U(x, 0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) = \sin\left(\frac{\pi x}{L}\right).$$

Comparing the two sides of the last equality, and given that the sine functions in the infinite series are independent of each other one finds that

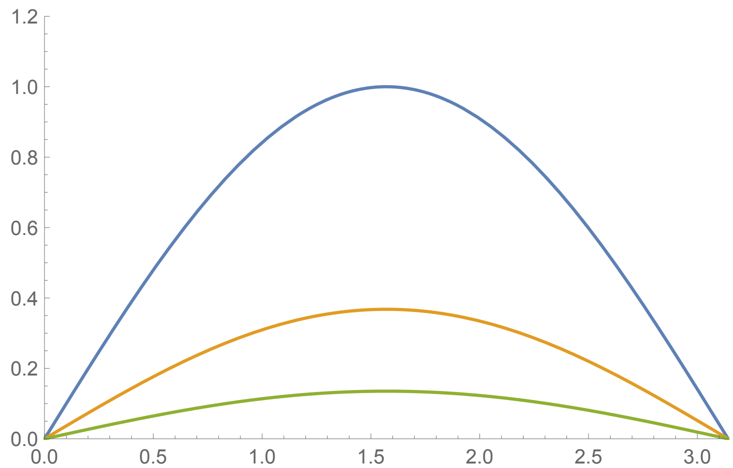
$$a_1 = 1, \qquad a_n = 0, \quad n \geq 2.$$

Thus, the particular solution to the heat equation is given by

$$U(x, t) = e^{-\pi^2 \varkappa t / L^2} \sin\left(\frac{\pi x}{L}\right).$$

A plot of the solution for various values of t is given below. Observe that

$$U(x, t) \to 0, \qquad \text{as} \quad t \to \infty.$$



Example 1.2. Let the initial conditions be given by

$$f(x) = 1, \qquad x \in [0, L].$$

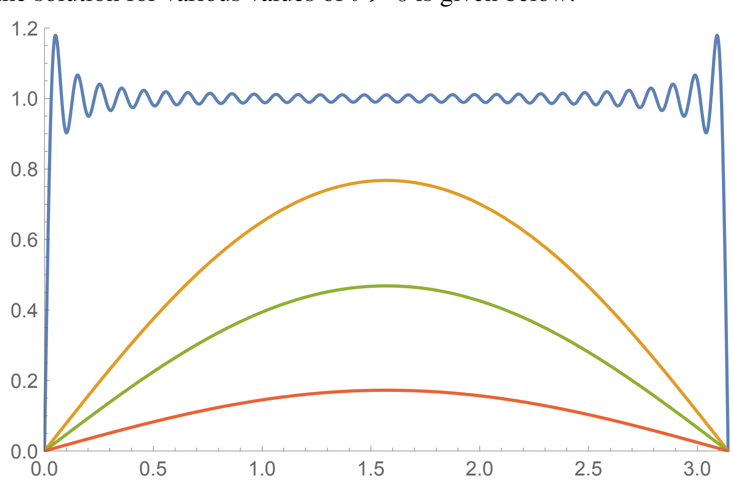
In this case we have to explicitly compute the Fourier coefficients —this is because the constant function does not appear in the series. One has that

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L \\ &= -\frac{2}{n\pi}\left((-1)^n - 1\right) \\ &= \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{n\pi} & n \text{ odd} \end{cases} \end{aligned}$$

Hence, one can write

$$U(x,t) = \frac{4}{\pi}\sum_{n \text{ odd}}^{\infty} e^{-n^2\pi^2\varkappa t/L^2}\sin\left(\frac{n\pi x}{L}\right).$$

A plot of the solution for various values of $t > 0$ is given below:



Observe that the solution for $t > 0$ instantly drops to 0 at the ends. Observe that for $n \geq 3$ one has that

$$e^{-\pi^2 \varkappa t/L^2} >> e^{-9\pi^2 \varkappa t/L^2}.$$

Thus, one has that

$$U(x,t) \approx \frac{4}{\pi} e^{-\pi^2 \varkappa t/L^2} \sin\left(\frac{\pi x}{L}\right).$$

In other words, the first term in the infinite series dominates.

Example 1.3. Let $L = 1$ and

$$f(x) = \begin{cases} 1 & 0 < x < 1/2 \\ 0 & 1/2 < x < 1 \end{cases}.$$

Again, we need to compute explicitly the Fourier coefficients. In this case we have

$$a_n = 2\int_0^{1/2} \sin(n\pi x)dx = -\frac{2}{n\pi}\cos(n\pi x)\Big|_0^{1/2} = -\frac{2}{n\pi}\left(\cos(\frac{n\pi}{2}) - 1\right).$$

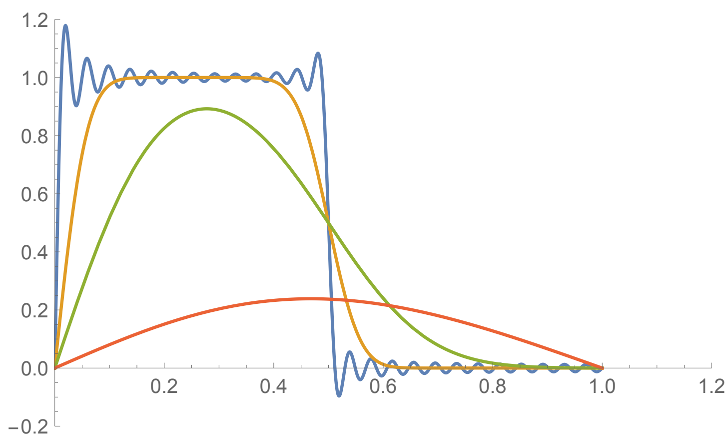
Hence,

$$U(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi/2}{n}\right) e^{-n^2\pi^2\varkappa t} \sin n\pi x.$$

Observe that

$$1 - \cos\frac{n\pi}{2} = 1,\ 2,\ 1,\ 0,\ 1,\ 2, \ldots.$$

A plot of the solution for various $t > 0$ is given below:



Observe that initially one has a step. The solution immediately becomes smooth. It gets more symmetric and sinusoidal as time increases.

Next, let's consider a mixed boundary condition problem for heat equations on an interval.

Example 1.4.

$$\begin{cases} U_t - U_{xx} = 0, x \in [0, \frac{\pi}{2}] \\ U(x,0) = 2\cos x \\ U_x(0,t) = 0, U(\frac{\pi}{2}, t) = 0. \end{cases}$$

Here we have the heat constant $\varkappa = 1$.

Suppose we have a separated variable solution $U(x,t) = X(x)T(t)$, we then get

$$X\dot{T} = X''T$$
$$\frac{\dot{T}}{T} = \frac{X''}{X} = -\lambda,$$

where λ is a constant. As before, this gives us 2 ODEs

$$X'' = -\lambda X$$
$$\dot{T} = -\lambda T.$$

Combing the ODE of X with the boundary conditions give us the following boundary value problem.

$$X'' = -\lambda X$$
$$X'(0) = 0, X(\frac{\pi}{2}) = 0.$$

As before, using the boundary conditions and integration by parts, we can show that the eigenvalues satisfy $\lambda > 0$ and so the general solutions for X is

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$
$$X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x.$$

Using the first boundary condition, we get

$$0 = X'(0) = 0 + B\sqrt{\lambda}.$$

So $B = 0$ and $A \neq 0$. Using the second boundary condition, we get

$$0 = X'(L) = A\cos\sqrt{\lambda}L.$$

So $\sqrt{\lambda}L = n\pi - \frac{1}{2}\pi$.

We get the eigenvalues are $\lambda_n = (2n-1)^2$, for $n = 1, 2\ldots$ and the eigenfunctions are

$$X_n(x) = \cos[(2n-1)x].$$

Knowing λ_n, we can go back to solve the ODE $\dot{T} = -\lambda_n T$ for T and get

$$T_n(t) = e^{-(2n-1)^2 t}.$$

Then general solutions are

$$U(x,t) = \sum_{n=1}^{\infty} a_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} a_n e^{-(2n-1)^2 t}\cos[(2n-1)x].$$

Next, we use the initial condition to specify the $a_n's$. When $t = 0$, we have

$$2\cos x = U(x,0) = \sum_{n=1}^{\infty} a_n \cos[(2n-1)x].$$

By the orthogonality of the $\cos[(2n-1)x]$ trigonometric functions, we "observe" that $a_n = 0$ except for $n = 1$. Moreover, the $n = 1$ term have to match and so $a_1 = 2$. Thus the solution to this question is

$$U(x,t) = 2e^{-t}\cos x.$$

2. Energy for heat equation on the interval and applications

Recall that the wave equation has an energy quantity that was preserved along time, which was useful in proving uniqueness of solutions to the wave equations. We can also define an energy quantity for the heat equation.

Consider the heat equation on the interval with fixed boundary condition from the last section

$$\begin{aligned}
&U_t = \varkappa U_{xx}, \qquad x \in [0, L], \quad t > 0,\\
&U(x, 0) = f(x),\\
&U(0, t) = 0, \qquad U(L, t) = 0.
\end{aligned}$$

We define the energy to be

$$E[U](t) = \frac{1}{2}\int_0^L U^2(x,t)dx.$$

Proposition 2.1. *The energy is non-increasing along time. It's preserved along time if and only if U is constant.*

Proof.

$$\begin{aligned}
\frac{d}{dt}E[U](t) =& \frac{d}{dt}\left[\frac{1}{2}\int_0^L U^2(x,t)dx\right]\\
=&\frac{1}{2}\int_0^L 2U(x,t)U_t(x,t))dx\\
=&\varkappa\int_0^L U(x,t)U_{xx}(x,t))dx\\
=&\varkappa U \cdot U_x|_0^L - \int_0^L \varkappa(U_x)^2dx\\
=&-\varkappa\int_0^L (U_x)^2dx\\
\leq& 0.
\end{aligned}$$

So the energy is non-increasing along time.

Since $U_x^2 \geq 0$, so the energy is preserved if and only if $U_x \equiv 0$.

If $U_x \equiv 0$, we have $U_{xx} \equiv 0$ and $U_t = \varkappa U_{xx} = \equiv 0$. So if the energy is preserved along time, U must be constant. □

By the Wirtinger inequality (we will use it here without proof): if f satisfies $f(0) = f(L) = 0$ on the interval $[0, L]$, we must have

$$\int_0^L [f^2(x)]dx \leq C_0 \int_0^L [f'(x)]^2 dx,$$

for some universal constant $C_0 > 0$. Using this we have the energy satisfies

$$\begin{aligned}\frac{d}{dt}E[U](t) \leq& -\varkappa\int_0^L (U_x)^2 dx\\ \leq& \frac{-\varkappa}{C_0}\int_0^L U^2 dx\\ =& \frac{-2\varkappa}{C_0}E[U](t).\end{aligned}$$

This gives the decay of energy

$$E[U](t) \leq E[U](0) \cdot e^{\frac{-2\varkappa}{C_0}t} \to 0,$$

as $t \to \infty$.

As an application of the non-increasing of energy, we show the uniqueness of solutions of heat equations on interval with initial and boundary conditions.

Theorem 2.2. *Let U_1 and U_2 are 2 solutions to the following heat equation an interval with initial and Dirichlet boundary conditions:*

$$\begin{aligned}&U_t = \varkappa U_{xx}, x \in [0, L]\\ &U(x, 0) = f(x)\\ &U(0, t) = h(t), U(L, t) = g(t).\end{aligned}$$

Then we must have $U_1 \equiv U_2$.

Proof. Let $V(x, t) = U_1(x, t) - U_2(x, t)$. Then by the principle of superposition, we have V satisfies the equation

$$\begin{aligned}&V_t = \varkappa V_{xx}, x \in [0, L]\\ &V(x, 0) = 0\\ &V(0, t) = 0, V(L, t) = 0.\end{aligned}$$

The energy at time $t = 0$ is $E[V](0) = 0$. By the non-increasing of energy Proposition 2.1 $\frac{d}{dt}E[V](t) \leq 0$ and the fact that $E[V](t) \geq 0$, we must then have

$$E[V](t) \equiv 0.$$

So $V \equiv 0$ for all $t \geq 0$ and $U_1 = U_2$. □

3. The heat equation on the real line

In this section we will see how to solve the problem

$$\begin{aligned}&U_t = \varkappa U_{xx}, \qquad x \in \mathbb{R}, \quad t > 0,\\ &U(x, 0) = f(x).\end{aligned}$$

That is, we want to solve the heat equation on the real line given that we know the initial form of U.

In order to solve this problem we will need some further assumptions on the solution $U(x, t)$ and the initial data $f(x)$. In particular, we want $U(x, t)$ to be absolutely integrable —that is,

$$\int_{-\infty}^{\infty} |U(x, t)| dx < \infty. \tag{3.1}$$

Also, we require that U and U_x go to zero at infinity —that is,

$$U(x,t),\, U_x(x,t) \longrightarrow 0, \qquad x \to \pm\infty.$$

Note. *Observe that to have condition* (3.1) *one needs* $U(x,t)$ *to go to zero at infinity.*

We also require f to be absolutely integrable:

$$\int_{-\infty}^{\infty} f(x)dx < \infty.$$

Example 3.1. Functions which are absolutely integrable are special —i.e. not all functions are absolutely integrable. Some examples are:

(i) $f(x) = \sin x$. One then has that

$$\int_{-\infty}^{\infty} |\sin x| dx = \infty.$$

That is, $\sin x$ is not integrable.

(ii) Let

$$f(x) = \frac{1}{1+x^2}.$$

One has that

$$\int_{-\infty}^{\infty} \left|\frac{1}{1+x^2}\right| dx = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \arctan x\Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi < \infty.$$

Thus $f(x) = 1/(1+x^2)$ is absolutely integrable.

(iii) Let $f(x) = e^{-x^2}$. From Calculus we know that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} < \infty,$$

so, again, absolutely integrable.

3.1. **Invariance properties of the heat equation.** An important property of the heat equation involves the behaviour of its solutions with respect to scalings of the coordinates. More precisely,

Lemma 3.2. *If* $U(x,t)$ *solves the heat equation then also* $V(x,t) \equiv U(ax, a^2t)$ *also solves the heat equation.*

Proof. Let $v = ax$, $w = a^2t$. Then, using the chain rule one finds that

$$\begin{aligned}
U_t(v,w) &= \frac{\partial w}{\partial t} U_w(v,w) = a^2 U_w(v,w),\\
U_x(v,w) &= \frac{\partial v}{\partial x} U_v(v,w) = a U_v(v,w),\\
U_{xx}(v,w) &= a^2 U_{vv}(v,w).
\end{aligned}$$

Hence,

$$U_t(ax,a^2t) - \varkappa U_{xx}(ax,a^2t) = a^2\Big(U_w(v,w) - \varkappa U_{vv}(v,w)\Big) = 0.$$

$\square$

Note. *Observe that*
$$\frac{v^2}{w} = \frac{a^2x^2}{a^2t} = \frac{x^2}{t}.$$
Thus, this hints that the ratio x^2/t is important for the heat equation. Thus, it makes sense to look for solutions of the form
$$U(x,t) = F\left(\frac{x^2}{t}\right).$$

In the following we will look for solutions with a slightly different form.

3.2. **Invariant solutions.** In this section we consider solutions to the heat equation of the form

$$U(x,t) = \frac{1}{t^{\alpha/2}} F\left(\frac{x}{\sqrt{t}}\right) \tag{3.2}$$

with α a constant to be determined. In view of the scaling property of solutions to the heat equation, the $U(x,t)$ as given by (3.2) satisfies the property
$$U(x,t) = \frac{1}{t^{\alpha/2}} U\left(\frac{x}{\sqrt{t}}, 1\right).$$
That is, solutions of the form (3.2) have **invariance properties**.

We now compute the partial derivatives of $U(x,t)$ as given by (3.2). For convenience, let
$$z \equiv \frac{x}{\sqrt{t}}.$$
Using the chain rule one finds that
$$\begin{aligned}
U_x(x,t) &= \frac{1}{t^{\alpha/2+1/2}} F'(z),\\
U_{xx}(x,t) &= \frac{1}{t^{\alpha/2+1}} F''(z),\\
U_t(x,t) &= -\frac{\alpha}{2t^{\alpha/2+1}} F(z) - \frac{z}{2t^{\alpha/2+1}} F'(z).
\end{aligned}$$
Thus, the heat equation gives
$$-\frac{\alpha}{2t^{\alpha/2+1}} F(z) - \frac{z}{2t^{\alpha/2+1}} F'(z) = \frac{\varkappa}{t^{\alpha/2+1}} F''(z).$$
So, if $t \neq 0$, the latter can be simplified to

$$\varkappa F''(z) + \frac{z}{2} F'(z) + \frac{\alpha}{2} F(z) = 0, \tag{3.3}$$

that is, we have obtained an ode for $F(z)$. In order to solve it, we need to fix the value of α. This requires making use of the extra requirements on $U(x,t)$ like absolute integrability —condition (3.1). In this relation it is noticed the following:

Lemma 3.3. *Let $U(x,t)$ be a solution to the heat equation which is absolutely integrable and satisfying $U_x(x,t) \to 0$ as $x \to \pm\infty$. Then*
$$\int_{-\infty}^{\infty} U(x,t)dx$$
is constant for $t \geq 0$.

Proof. To see this integrate the heat equation over the real line:

$$\int_{-\infty}^{\infty} U_t(x,t)dx = \varkappa \int_{-\infty}^{\infty} U_{xx}(x,t)dx.$$

This can be rewritten, using the Fundamental Theorem of Calculus as

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty} U(x,t)dx &= \varkappa U_x(x,t)\Big|_{-\infty}^{\infty} \\ &= 0.\end{aligned}$$

The last equality follows from the requirement that U_x goes to zero at infinity. Thus, the integral

$$\int_{-\infty}^{\infty} U(x,t)dx$$

does not depend on t —that is, it is constant. □

Note. *Without loss of generality we can set*

$$\int_{-\infty}^{\infty} U(x,t)dt = 1. \tag{3.4}$$

Recalling that $U(x,t) = t^{-\alpha/2}F(x/\sqrt{t})$ it follows then from equation (3.4) that

$$\begin{aligned}1 &= \frac{1}{t^{\alpha/2}}\int_{-\infty}^{\infty} F\left(\frac{x}{\sqrt{t}}\right)dx \\ &= \frac{\sqrt{t}}{t^{\alpha/2}}\int_{-\infty}^{\infty} F(z)dz,\end{aligned}$$

where in the last line we have used the change of coordinates $z = x/\sqrt{t}$. Thus, setting

$$\alpha = 1$$

we obtain

$$\int_{-\infty}^{\infty} F(z)dz = 1. \tag{3.5}$$

With this choice of the constant α equation (3.3) reduces to

$$\varkappa F'' + \frac{z}{2}F' + \frac{1}{2}F = 0.$$

One can readily check that

$$\begin{aligned}\varkappa F'' + \frac{z}{2}F' + \frac{1}{2}F &= \varkappa F'' + \frac{1}{2}\big(zF\big)' \\ &= \left(\varkappa F' + \frac{z}{2}F\right)' = 0.\end{aligned}$$

Hence, integrating one obtains

$$\varkappa F' + \frac{z}{2}F = C = \text{constant}.$$

To determine the constant C it is observed that in order for equation (3.5) to make sense one needs that

$$F(z),\ F'(z) \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \pm\infty.$$

Thus, in fact, one has that

$$C = 0$$

and the differential equation reduces to

$$\varkappa F' + \frac{z}{2}F = 0.$$

This is an equation that can be readily solved by means separation. Writing it in the form

$$\frac{dF}{dz} = -\frac{z}{2\varkappa}F$$

one then has that

$$\int \frac{dF}{F} = -\frac{1}{2\varkappa}\int z dz + \tilde{C},$$

with $\tilde{C}$ an integration constant. Writing, for convenience, the integrating constant as $\ln \tilde{C}$ one obtains

$$\ln F = -\frac{1}{4\varkappa}z^2 + \ln \tilde{C},$$

so that

$$F(z) = \tilde{C}e^{-z^2/4\varkappa}.$$

To determine the integration constant we recall, again, the normalisation condition (3.5). It follows then that

$$\begin{aligned} 1 &= \tilde{C}\int_{-\infty}^{\infty} e^{-z^2/4\varkappa}dz \\ &= \tilde{C}\int_{-\infty}^{\infty} e^{-y^2}2\sqrt{\varkappa}dy \\ &= 2\tilde{C}\sqrt{\varkappa\pi}, \end{aligned}$$

where in the second line we have used the substitution $y = z/2\sqrt{\varkappa}$. Hence,

$$\tilde{C} = \frac{1}{\sqrt{4\varkappa\pi}},$$

so that

$$F(z) = \frac{1}{\sqrt{4\varkappa\pi}}e^{-z^2/4\varkappa}.$$

Recalling the ansatz (3.2) one finally finds that

$$U(x,t) = \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}}.$$

This solution is known as the *heat kernel* or *fundamental solution* of the heat equation. We denote it by:

$$K(x,t) = \frac{e^{-\frac{x^2}{4\varkappa t}}}{\sqrt{4\varkappa\pi t}}.$$

We note the following properties:

(i) By construction $K(x,t)$ satisfies the heat equation. That is,

$$K_t = \varkappa K_{xx}, \qquad x \in \mathrm{R}, \quad t > 0.$$

(ii) The heat kernel is an even function —that is, $K(x,t) = K(-x,t)$.

(iii) $K(x,t)$ is a smooth function (i.e. C^∞) for $x \in \mathbb{R}$, $t > 0$.

(iv) One has that
$$\int_{-\infty}^{\infty} K(x,t)dx = 1, \qquad t \geq 0.$$
(v) For $x \neq 0$ one has that
$$K(x,t) \to 0, \qquad \text{as} \quad t \to 0^+,$$
while for $x = 0$ one has that
$$K(0,t) \to \infty, \qquad \text{as} \quad t \to 0^+.$$
(vi) For any $x \in \mathbb{R}$, one has
$$K(x,t) \to 0, \qquad \text{as} \quad t \to \infty$$

Remark 3.4. Properties *(i)-(iv)* above, follow from the construction of the heat Kernel given in the previous section. Only property *(v)* requires further work. If $x \neq 0$ then to compute the limit it is enough to consider
$$\frac{e^{-1/t}}{\sqrt{t}} = \frac{1}{\sqrt{t}e^{1/t}} \longrightarrow 0 \qquad \text{as} \quad t \to 0^+,$$
given that $e^{1/t} \to \infty$ and recalling that the exponential grows faster than any power of t so it dominates $\sqrt{t}$. For $x = 0$ one has that
$$K(0,t) = \frac{e^0}{\sqrt{4\pi\varkappa t}} = \frac{1}{\sqrt{4\pi\varkappa t}} \longrightarrow \infty \qquad \text{as} \quad t \to 0^+.$$

Note. *Property* (v) *together with* (iv) *show that* $K(x,0)$ *is a very special object —in fact, it turns out that* $K(x,0)$ *cannot be a function. It is a more general type of object known as* generalised function *or* distribution*.*