DYNAMICAL SYSTEMS

MTH744U/P

SEMESTER A 2023-24

WEEKS 9, 10, 11, 12

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Weeks 9 - 12

Week 9 – Non-linear systems

WK9.1

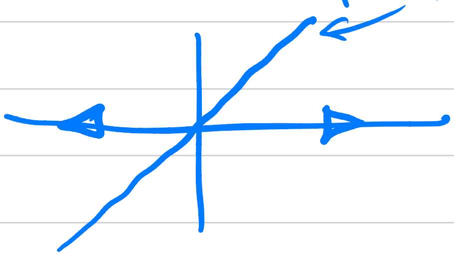
Gain insight into the nature of non-linear systems on $\mathbb{R}^2$, the real plane.

$$\underline{\dot{x} = f(x,y)}, \quad \underline{\dot{y} = g(x,y)}, \quad f, g : \mathbb{R}^2 \to \mathbb{R}$$

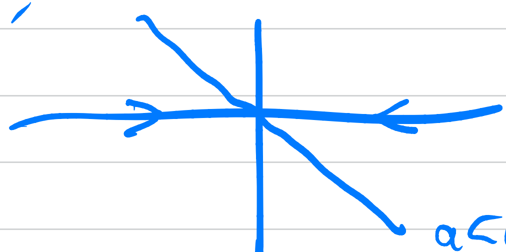
not necessarily linear from now on! $f(x,y)$ more than $ax + by$.

Reconsider systems on $\mathbb{R}$: $\dot{x} = f(x)$, $f : \mathbb{R} \to \mathbb{R}$

f linear : $f(x) = ax$, $a \in \mathbb{R}$. $\dot{x} = ax$.



$a > 0$



$a = 0$

$a < 0$

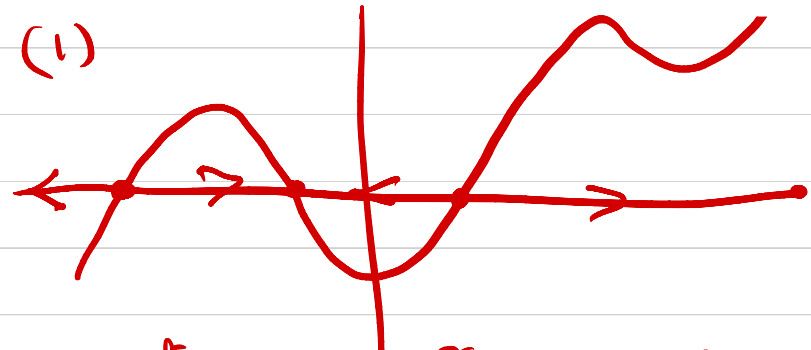
WK9.2

f non-linear

(i) graph of f

(ii) Linear stability

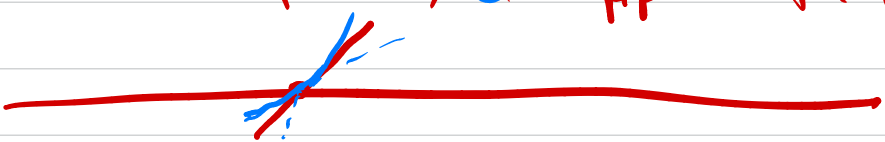
(1)



Given f, has zeroes at $x_1, x_1, x_2, x_3, \ldots$

Linear approximation of f at $x = x_1$

Calculate $f'(x_1)$ ① Suppose $f'(x_1) > 0$ ✓

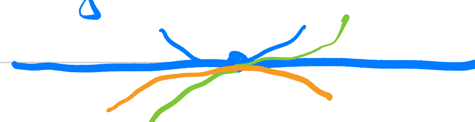


$x = x_1$

② Also $f'(x_1) < 0$ ✓

stable

③ $f'(x_1) = 0$, further investig



Wk9.3

Corresponding program for $\mathbb{R}^2$

- Locate fixed points $\dot{x} = \dot{y} = 0$ i.e. $f(x,y) = g(x,y) = 0$.

  Find roots of $f(x,y) = g(x,y) = 0$

  - suppose $(x,y) = (x^*, y^*)$ is a solution.

- Linearised system at a fixed point $(x,y) = (x^*, y^*)$ $(= \underline{x}^*)$?

- Taylor expansion

$$\dot{x} = f(x,y) = \cancel{f(\underline{x}^*)} + \overset{u}{(x - x^*)} \frac{\partial f}{\partial x}(\underline{x}^*) + \overset{v}{(y - y^*)} \frac{\partial f}{\partial y}(\underline{x}^*) + \cdots$$

$$\dot{y} = g(x,y) = \cancel{g(\underline{x}^*)} + \overset{u}{(x - x^*)} \frac{\partial g}{\partial x}(\underline{x}^*) + \overset{v}{(y - y^*)} \frac{\partial g}{\partial y}(\underline{x}^*) + \cdots$$

$u = x - x^*$ $\quad \dot{u} = \dot{x}$

$v = y - y^*$ $\quad \dot{v} = \dot{y}$

Extract linear system (Jacobian Matrix)

Introduce local coordinates $u = x - x^*$, $v = y - y^*$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x}(\underline{x}^*) & \frac{\partial f}{\partial y}(\underline{x}^*) \\ \frac{\partial g}{\partial x}(\underline{x}^*) & \frac{\partial g}{\partial y}(\underline{x}^*) \end{bmatrix}}_{\text{Jacobian matrix}} \begin{bmatrix} u \\ v \end{bmatrix} + \text{h.o.t.}$$

Hyperbolic linear systems - systems $\dot{\underline{x}} = A\underline{x}$, whose phase portrait remains robust under sufficiently small perturbations of the coeffs of A. (qualitatively the same).

Not hyperbolic.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} \qquad \lambda$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \lambda^2 + 1 = 0 \qquad \lambda = \pm i$$

WK9.5

$$A = \begin{bmatrix} \varepsilon & -1 \\ 1 & \varepsilon \end{bmatrix} \qquad \lambda = \varepsilon \pm i$$

Reminder

$$J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$\dot{r} = \alpha r \qquad \dot{\theta} = \beta.$$

$\downarrow$

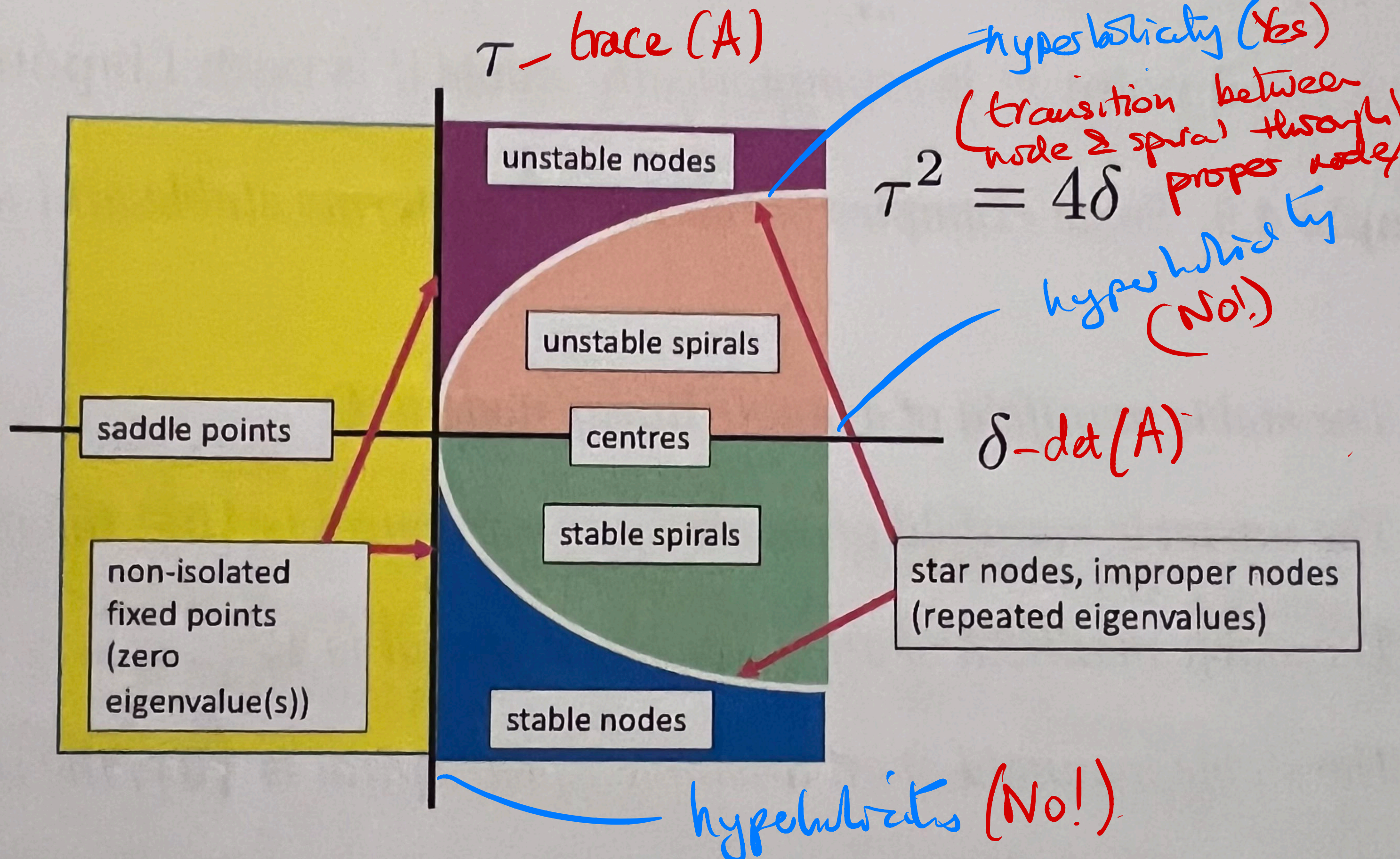
unstable spiral $\varepsilon > 0$

centre. $\varepsilon = 0$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \lambda_1 = 1, \ \lambda_2 = 2$$

small change in $\lambda_1 \approx 1$, $\lambda_2 \approx 2$.

WK9.6



τ – trace (A)
hyperbolicity (Yes)
(transition between node & spiral through proper node)
$\tau^2 = 4\delta$
hyperbolicity (No!)
unstable nodes
unstable spirals
saddle points
centres
δ – det (A)
stable spirals
non-isolated fixed points (zero eigenvalue(s))
star nodes, improper nodes (repeated eigenvalues)
stable nodes
hyperbolicity (No!)

WK9.7

Hartman-Grobman Theorem (p36)

A linear system $\dot{\underline{x}} = \underline{A}\,\underline{x}$ is st.b. hyperbolic if none of the eigenvalues has zero real part

Thm 5.2 (p36) The qualitative behaviour of a dynamical system and its linearised system on a sufficiently small nbd of a fixed point are the same provided the Jacobian is hyperbolic

Examples of non-linear systems with non-hyperbolic fixed points

WK9.8

Ex 5.1 (p37)

$$\dot{x} = \underset{f}{y - x^3}, \quad \dot{y} = \underset{g}{-x - y^3}$$

$x = y = 0$, fixed point.

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{\underline{0}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

λ_1, λ_2? $\quad \lambda^2 + 1 = 0$, $\quad \lambda = \pm i$. $\quad \lambda_1 = +i$, $\lambda_2 = -i$

$\dot{x} = y$, $\quad \dot{y} = -x$ $\qquad x^2 + y^2 =$ conserved.

(r is conserved)

$$dt = \frac{dx}{y} = \frac{dy}{-x}$$

WK9.9

$\dot{x} = y - x^3, \quad \dot{y} = -x - y^3$

$r\dot{r} = x\dot{x} + y\dot{y} = \cancel{xy} - x^4 - \cancel{xy} - y^4$

$= -(x^4 + y^4)$ $\dot{r} > 0 \ (r > 0)$

$(x^4 + y^4 > 0)$

$r > 0 \quad \dot{r} < 0$

stable spiral NL

Centre L

$\underline{r^2\dot{\theta} = x\dot{y} - y\dot{x}}$

$r^2\dot{\theta} = x\dot{y} - y\dot{x} \Rightarrow r^2\dot{\theta} = x(-x - y^3) - y(y - x^3).$

$= -x^2 - y^2 - xy^3 + yx^3$

$\dot{\theta} = -1 + r^2(\sin\theta\cos\theta)(\cos^2\theta - \sin^2\theta)$

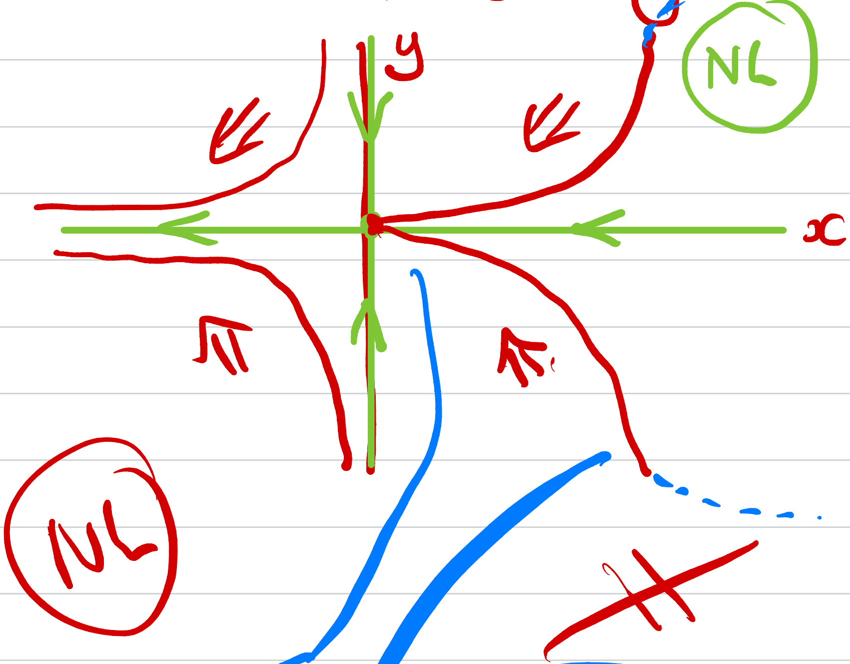
$= -1 + r^2 \frac{\sin 2\theta}{2} \cdot \cos 2\theta$

$= -1 + r^2 \frac{\sin 4\theta}{4}$

Calculation added

for $|r| < 4$ $\dot{\theta} < 0$ (clockwise rotation)

$\dot{x} = -x^2, \quad \dot{y} = -y$



tangentially correct!

topologically correct!

$$J = \begin{bmatrix} -2x & 0 \\ 0 & -1 \end{bmatrix}_{0} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\left.\begin{aligned} \dot{x} &= 0 \\ \dot{y} &= -y. \end{aligned}\right\}$$

L

Wk 9.11

Ex 5.3

$\dot{x} = y - xy^2 , \quad \dot{y} = -x + yx^2$

$J(\underline{0}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ — centre. Linear. S

$r\dot{r} = xy - x^2y^2 - xy + y^2x^2 = 0$

$r > 0 \quad \dot{r} = 0$ — concentric. circles

non-linear. centre

Here linear centre <u>and</u> non-linear centre

Ex 5.5 $\dot{x} = -x + x^3, \quad \dot{y} = -2y$

FPs $-x + x^3 = 0 \qquad -2y = 0$

$x(x^2 - 1) = 0 \qquad y = 0$

$x = 0, \pm 1$

$x_1^* = (0,0) \quad x_2^* = (1,0) \quad x_3^* = (-1,0)$

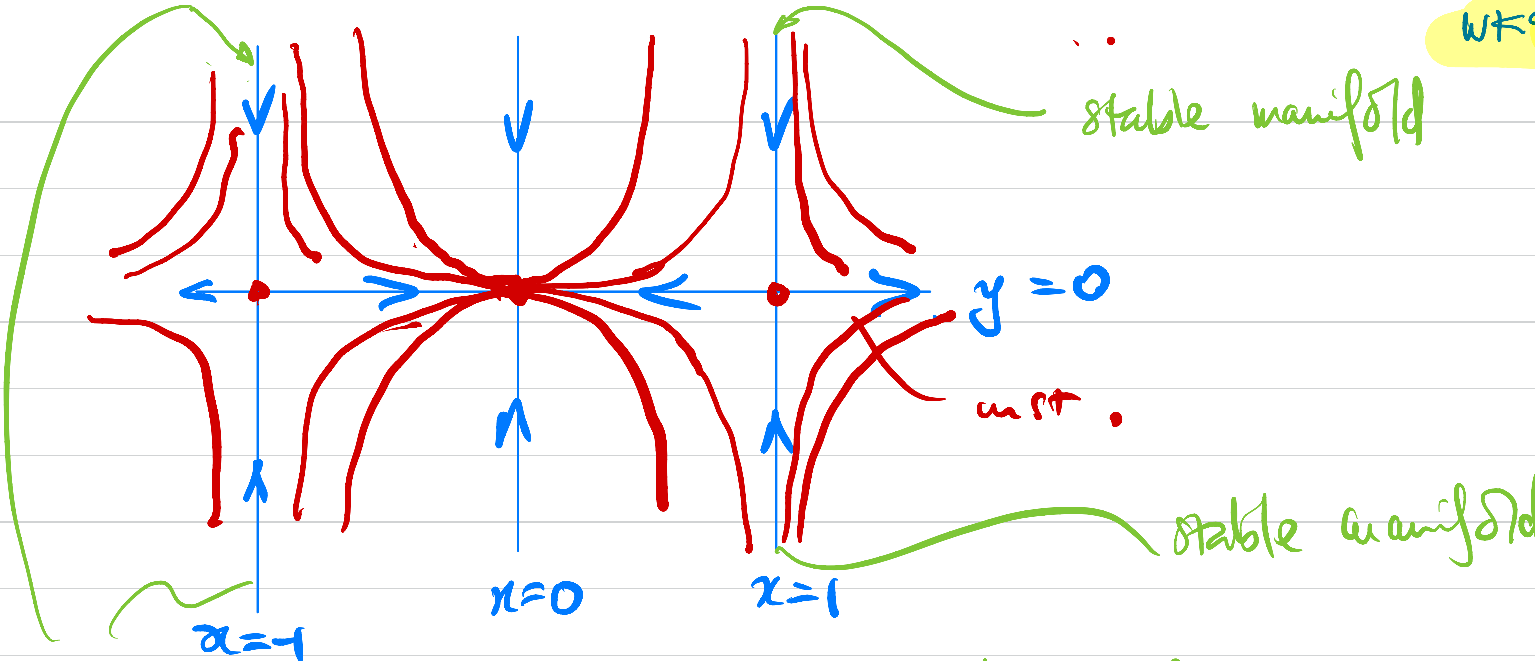
$$J = \begin{bmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\underline{x_1^*}$ $J = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ ✓ $\lambda_1 = -1$, $\lambda_2 = -2$

$\underline{x_2^*}$ $J(x_2^*) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ ✓

x_3^* $J(x_3^*) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ ✓

WK9.13



stable manifold

All points (x,y) with $|x|<1$ form the basin of attraction of the origin.

WEEK 10

Ex 5.5 p38 Phase portrait of 5.12 (decoupled / "rectangular")

Ex 5.6 p39 Phase portrait of 5.14 (non-decoupled!)

$$\dot{x} = y, \quad \dot{y} = x - x^3$$

Three fixed pts ??

$y = 0$, $x = 0, \pm 1$ $\quad \underline{x}_1^* = (0,0)$ $\underline{x}_2^* = (1,0)$, $x_3^* = (-1,0)$

Jacobians: $J = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix} \rightarrow ???$ $\quad \lambda^2 + 2 = 0$

x_1^* - saddle type, x_2^*, x_3^* (linear centre type)

- also non-linear centre!??

WK 10.2

Integral curves?

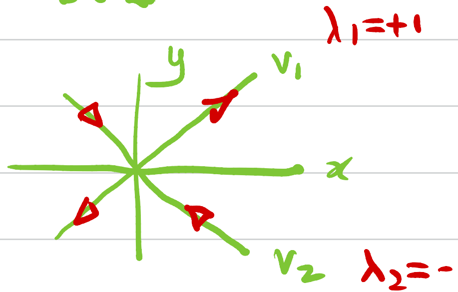
Eigenvectors at $(0,0)$ $\quad \underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

? ?

$\lambda_1 = \pm 1$? $\quad \lambda_2 = \pm 1$

$\underline{\lambda_1 = +1}$ $\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = +1 \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow y = x \quad \underline{v}_1$

$\underline{\lambda_2 = -1}$ $\quad \underline{v}_2$



$\dot{x} = y, \quad \dot{y} = x - x^3$

$$\frac{dx}{y} = \frac{dy}{x - x^3} \cancel{= dt}$$

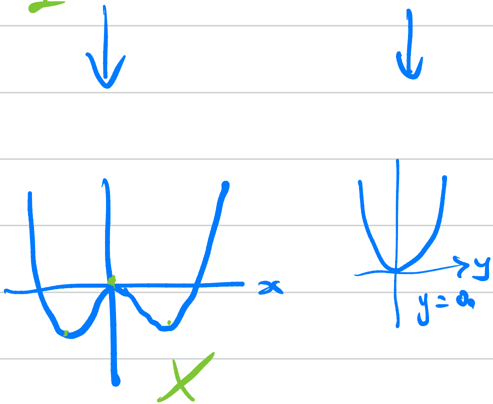
$$\int x - x^3 \, dx = \int y \, dy$$

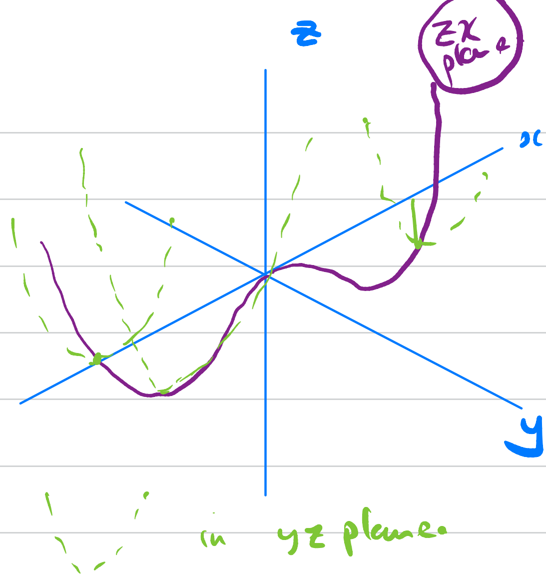
$$\frac{y^2}{2} + \frac{x^4}{4} - \frac{x^2}{2} = \text{Const.}$$

$$y^2 + \frac{x^4}{2} - x^2 = \text{Const.}$$

$$V(x,y) = y^2 + \frac{x^4}{2} - x^2.$$

$$\frac{x^4}{2} - x^2 + y^2.$$





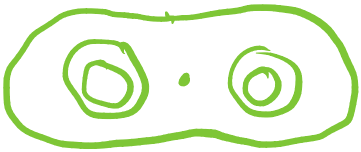
So by using the HGLT we cannot be sure of the non-linear status of the fixed points at $(1,0)$ and $(-1,0)$. Fortunately we can find integral curves for this system if we reconfigure the system equations 5.14 as

$$dt = \frac{dx}{y} = \frac{dy}{x - x^3} \tag{5.16}$$

from which we get, by separating variables,

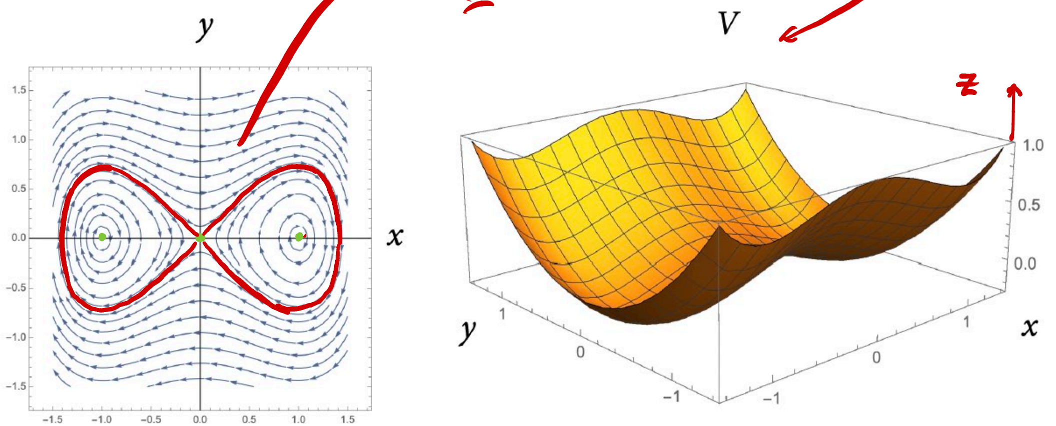
$$V(x,y) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2} = constant \tag{5.17}$$

i.e. the quantity V is conserved by the system 5.14 which means integral curves of the system are confined to the level curves of V. Of course, V = constant curves are equivalent to contour lines or level curves (i.e z = constant of the surface $z = V(x,y)$ in 3-dimensional xyz-space.



saddle connection
unstable manifold
= stable manifold

$z = V(x,y)$



(a) 1 unstable
2 centres (Lin & Non-lin)

(b) No Asymp Stable points

Figure 21: (a) The phase portrait for the system 5.23 follows the level curves of the first integral V. (b) Note the 'Mathematica' picture of the surface $z = V(x,y)$ does not capture the saddle point at the origin and its unstable/stable manifolds, but it does show up the non-linear centres at $(\pm 1, 0)$ well.

Example
Check $\dot{x} = 4y, \ \dot{y} = -x + x^2$

Wk 10.3

Fixed points $x = 0, 1$, $y = 0$ so

$x_1^* = (0,0)$, $x_2^* = (1,0)$

Jacobian $= \begin{bmatrix} 0 & 4 \\ -1+2x & 0 \end{bmatrix}$ at (x,y)

$\underline{x_1^*}$ $J = \begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda = \pm 2i$ (linear centre)

$\underline{x_2^*}$ $J = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_1, \lambda_2 = \pm 2$ (saddle)

WK 10.4

Integral curves?

$$dt = \frac{dx}{y} = \frac{dy}{x^2-x} \Rightarrow \int x^2-x\,dx = \int y\,dy$$

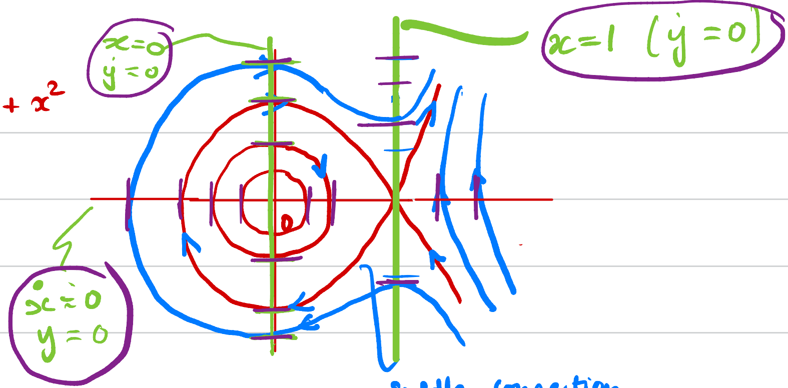
$$\Rightarrow \frac{x^3}{3} - \frac{x^2}{2} = \frac{y^2}{2} + C$$

Check eigenvectors etc. for $(0,0)$ & $(1,0)$

Consider $V(x,y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^3}{3}$

Suggest the nature of the curves $V(x,y) = \text{constant}$

$\dot{x} = y$
$\dot{y} = -x + x^2$



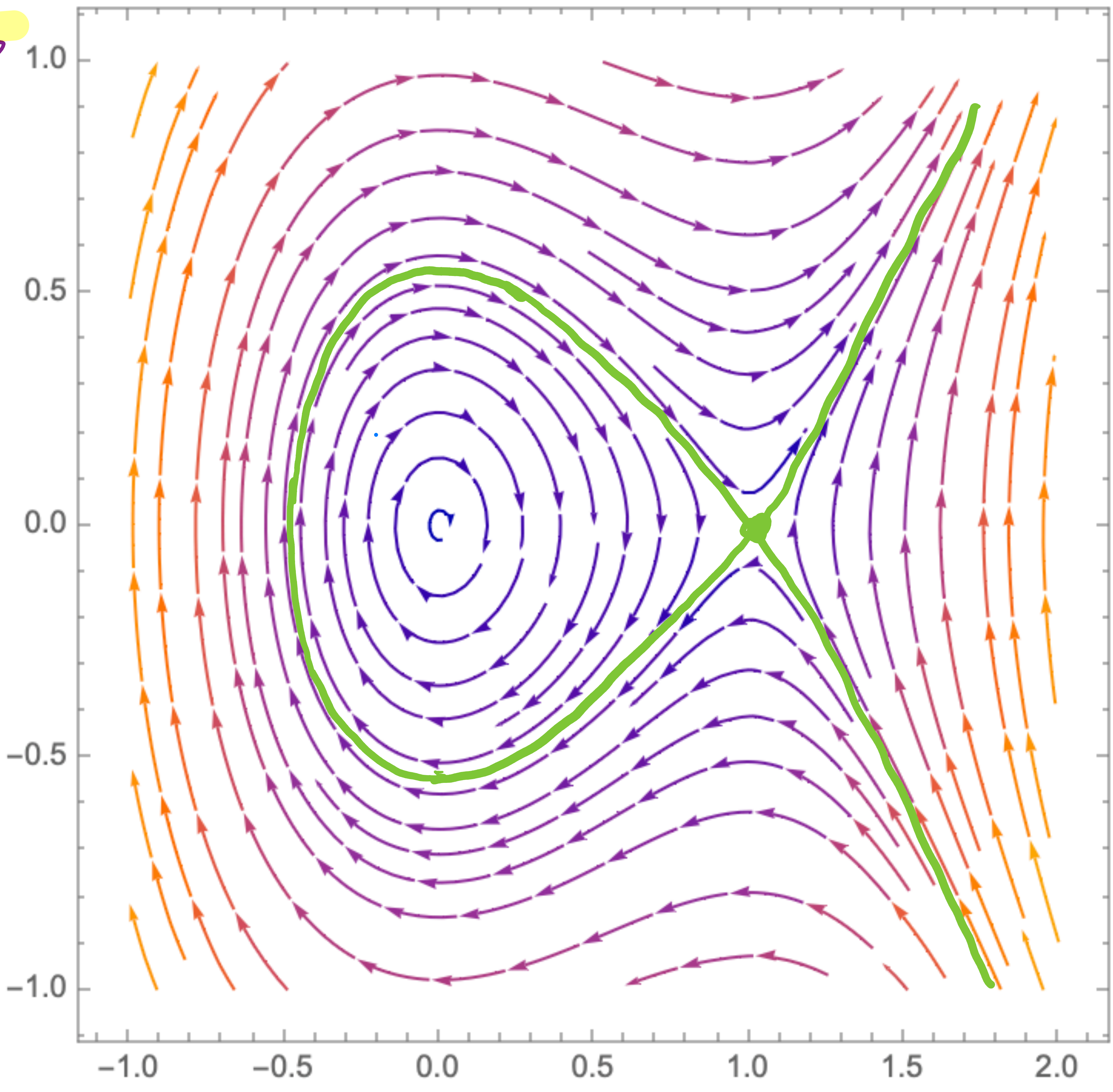
Saddle connection
- arises from conservative system.

See actual streamplot (Mathematica) on next page.

```
StreamPlot[{y, -x + x^2}, {x, -1, 2}, {y, -1, 1}]
```

WK10.b

Out[4]=



5.4 Conservative and gradient systems

WK 10.7

A real-valued function $H : \mathbb{R}^2 \to \mathbb{R}$ is a *constant of the motion* or *first integral* of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, and $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ if H is constant for any solution curve, i.e. $H(\underline{x}(t)) \equiv H(\underline{x}(0))$ for all $t \in \mathbb{R}$. The "trivial constant" of the motion that $H(x, y) \equiv C$, a constant, on an *open* set in $\mathbb{R}^2$ is not allowed as it offers no information on the nature of the solution curves.

$H(x(t), y(t)) = H(x(0), y(0))$

For $H(x, y)$ to be a constant of the motion, we require

$\underline{x}(t) = (x(t), y(t))$

$$\frac{d}{dt}(H(\underline{x}(t)) \equiv 0,$$

which implies

$$\frac{d}{dt}(H(\underline{x}(t)) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} \equiv 0.$$

5.4.1 Conservative systems

WK 10.9

Newton's second law of motion, (II), states that the force applied to an object is proportional to its acceleration, with the constant of proportionality being the mass of the object. This law can be written as an ODE in the form

$\ddot{x} = \frac{d^2x}{dt^2}$.

$$m\ddot{x} = F(x)$$

2nd order, 1 variable

where x is the position coordinate, m is the mass, and $F(x)$ is the force applied at the position x. Converting this second order ODE into a first order equation, we obtain

$\dot{\underline{x}} = \underline{f}(\underline{x})$

$$\dot{x} = y; \; \dot{y} = \frac{1}{m}F(x).$$

1st order in 2 var.

Define the function $E : \mathbb{R}^2 \to \mathbb{R}$ by

KE PE

$$E(x, y) = \frac{1}{2}my^2 + V(x), \tag{5.22}$$

where

$\underline{x}(t) = (x(t), y(t))$

$$V(x) = -\int F(x)dx.$$

Now

$$\frac{d}{dt}(E(\underline{x}(t)) = \frac{\partial E}{\partial x}\dot{x} + \frac{\partial E}{\partial y}\dot{y} = -F(x)y + my\dot{y} = y(-F(x) + m\ddot{x}) \equiv 0,$$

by Newton II.

The function E given in 5.22, the *energy*, is a constant of the motion for Newton II. Energy is conserved in this system - i.e. it is a *conservative system*. The energy E is seen as being comprised of two components: $\frac{1}{2}my^2$ is the *kinetic* energy; $V(x)$ is the *potential* energy

WK10.9

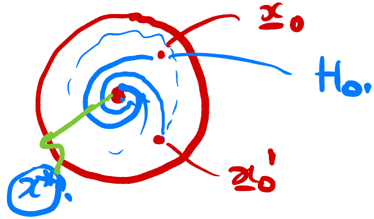
Theorem 5.3. *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, be a conservative system, with constant of motion H, then the system has no attracting points.*

i.e. no points which are asymptotically stable - locally attracting.

Proof. If there were to exist a neighbourhood U of a fixed point $\mathbf{x}^*$ of the system for which every solution $\mathbf{x}(t)$ with $\mathbf{x}_0 \in U$ satisfied $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$, then by continuity of H, $H(\mathbf{x}(t)) \to H(\mathbf{x}^*)$, as $t \to \infty$.

Since $H(\mathbf{x}(t))$ is constant as t varies, it follows that $H(\mathbf{x}(0)) = H(\mathbf{x}^*)$ on the neighbourhood U of x^*, i.e. $H(\mathbf{x}) = H(\mathbf{x}^*)$ for all $x \in U$ and is therefore a *trivial* constant of the motion.

H remains constant on the orbit $(x(t), y(t))$



Suppose $(x(t), y(t)) \to \underline{x}^*$

then $H(x(t), y(t)) (= H(x(0), y(0))) \to H(x^*)$

$\Rightarrow$ trivial constant of motion

$H(\underline{x}(t)) = H(x(0)) (= H(x^*)) \ \forall \ x(t) \in U$

Example 5.8. *Consider a Newton II system with potential energy $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$. The system we obtain using the equation 5.22 is*

$$\dot{x} = y, \ \dot{y} = x - x^3 \left(= -\frac{\partial V}{\partial x} \right), \tag{5.23}$$

which we considered earlier in equation 5.14. We have chosen $m = 1$ as its numerical value does not change the qualitative behaviour of the system. The system has fixed points at $(0, 0)$ - a saddle, and $(\pm 1, 0)$, are both centres. See figure 21.

5.4.2 Gradient systems

WK10. 10

Consider a differentiable function $F : \mathbb{R}^2 \to \mathbb{R}$. The *gradient system* with potential function F is

$$\dot{\mathbf{x}} = -\nabla F(\mathbf{x}), \tag{5.24}$$

where $\nabla F(\mathbf{x}) = \left(\frac{\partial F}{\partial x}(\mathbf{x}), \frac{\partial F}{\partial y}(\mathbf{x})\right)$. The fixed points of the gradient system are precisely those points for which $\nabla F(\mathbf{x}) = \mathbf{0}$, the so called *critical points* of F.

$\dot{x} = -\frac{\partial F}{\partial x}$

$\dot{y} = -\frac{\partial F}{\partial y}$

FPs ~ critical pts of F

max, min, saddle

Gradient & conservative systems are orthogonal!

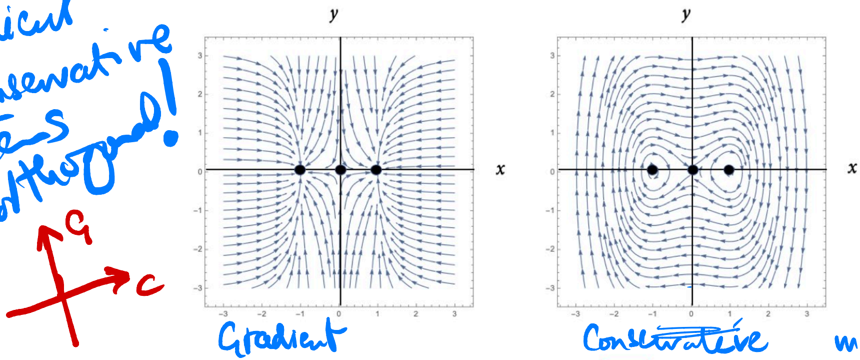


Figure 24: *The phase portraits of (a) the gradient system 5.24 and (b) the corresponding conservative system 5.23 with the same potential function* $E(x,y) = \frac{1}{2}my^2 + \frac{1}{4}x^4 - \frac{1}{2}x^2$.

Theorem 5.4. *A gradient system has no periodic orbits of positive period $T > 0$.*

Proof. If such a periodic orbit $\mathbf{x}(t)$ for $0 \le t \le T$ existed for the system 5.24, then the change in the value ΔF of F would be zero since $F(\mathbf{x}(0)) = F(\mathbf{x}(T))$, given $\mathbf{x}(0) = \mathbf{x}(T)$. But

$$\Delta F = \int_0^T \frac{dF}{dt} dt = \int_0^T \nabla F(\mathbf{x}).\frac{d\mathbf{x}(t)}{dt} dt = -\int_0^T ||\dot{\mathbf{x}}||^2 dt < 0.$$

which provides a contradiction.

We should note that this is the very opposite (or, perhaps, *orthogonal*!) to the behaviour of a conservative system.

We should note that this is the very opposite (or, perhaps, *orthogonal*!) to the behaviour of a conservative system.

WK 10.11

Example 5.9. *Prove that a conservative system with energy $E(x, y) = \frac{1}{2}my^2 + V(x)$, and a gradient system $\dot{\mathbf{x}} = -\nabla E(\mathbf{x})$, with the same energy E have orthogonal trajectories in their respective phase portraits, i.e. the vectors fields of the two systems are mutually orthogonal (hint: expand $\frac{dE}{dt}$*

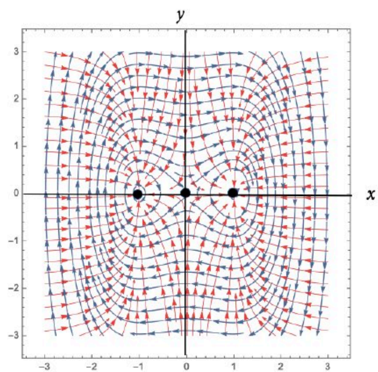


Figure 25: *The conservative (blue) and gradient (red) systems with the same potential, E - illustrated separately in figure 24 - are superimposed to show the mutual orthogonality of the flows (the dot product of the vector fields at every point of $\mathbb{R}^2$ is zero).*

WK10.12

and interpret the expression as a dot product of vectors). Note that the gradient system is now seen in the context of the "total" potential energy being $E(x, y)$*, not just the potential energy* $V(x, y)$ *of the mixed energy* $E(x, y)$ *in equation* 5.22.

Proof. *Given the potential energy function* $V(x)$*, the total energy function for the corresponding conservative system is*

$$E(x, y) = \frac{1}{2}my^2 + V(x),$$

cf. 5.22 *with system equations*

$$\dot{x}_C = y, \; \dot{y}_C = -\frac{V'(x)}{m}$$

Newtonian system

The corresponding gradient system for the energy function $E(x, y)$ *has the form*

$$\dot{x}_G = -\frac{\partial E}{\partial x} = -V'(x); \; \dot{y}_G = -\frac{\partial E}{\partial y} = -my.$$

Gradient system

It follows that the dot product of the two vector fields is

$$(\dot{x}_G, \dot{y}_G).(\dot{x}_C, \dot{y}_C) = \dot{x}_G.\dot{x}_C + \dot{y}_G.\dot{y}_C = (-V'(x)).y + (-my).(-\frac{V'(x)}{m}) \equiv 0. \qquad (5.25)$$

othogonal
⇉ vector fields
and flows.

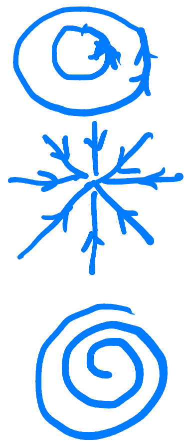
Therefore, the conservative and gradient vector fields are orthogonal, see figure 25. *This means that gradient systems follow the lines of maximum slope which are always perpendicular to level curves, a good direction to avoid when walking down a mountain!*

We should note that this is the very opposite (or, perhaps, *orthogonal*!) to the behaviour of a conservative system.

WK10.13

Example 5.9. *Prove that a conservative system with energy $E(x,y) = \frac{1}{2}my^2 + V(x)$, and a gradient system $\dot{\mathbf{x}} = -\nabla E(\mathbf{x})$, with the same energy E have orthogonal trajectories in their respective phase portraits, i.e. the vectors fields of the two systems are mutually orthogonal (hint: expand $\frac{dE}{dt}$*

Conservative and Gradient systems are extremes, "no potential loss vs. max potential loss"



C.

G

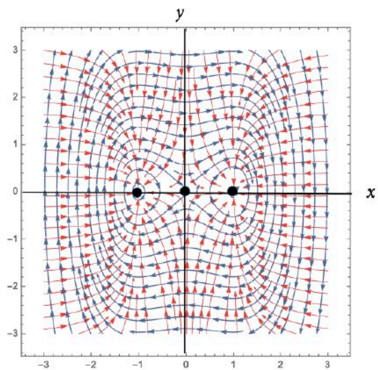


Figure 25: *The conservative (blue) and gradient (red) systems with the same potential, E - illustrated separately in figure 24 - are superimposed to show the mutual orthogonality of the flows (the dot product of the vector fields at every point of $\mathbb{R}^2$ is zero).*

WK10.4 **Theorem 5.3.** *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^2$, be a conservative system, with constant of motion H, then the system has no attracting points.* (no asymptotically stable fixed pts.

Proof. If there were to exist a neighbourhood U of a fixed point $\mathbf{x}^*$ of the system for which every solution $\mathbf{x}(t)$ with $\mathbf{x}_0 \in U$ satisfied $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$, then by continuity of H, $H(\mathbf{x}(t)) \to H(\mathbf{x}^*)$, as $t \to \infty$.

Since $H(\mathbf{x}(t))$ is constant as t varies, it follows that $H(\mathbf{x}(0)) = H(\mathbf{x}^*)$ on the neighbourhood U of x^*, i.e. $H(\mathbf{x}) = H(\mathbf{x}^*)$ for all $x \in U$ and is therefore a *trivial* constant of the motion.

Already discussed.

Example 5.8. *Consider a Newton II system with potential energy $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$. The system we obtain using the equation 5.22 is*

$$\dot{x} = y, \; \dot{y} = x - x^3 \; \left(= -\frac{\partial V}{\partial x}\right), \tag{5.23}$$

which we considered earlier in equation 5.14. We have chosen $m = 1$ as its numerical value does not change the qualitative behaviour of the system. The system has fixed points at $(0,0)$ - a saddle, and $(\pm 1, 0)$, are both centres. See figure 21.

[Note: the number r' in (ii) is the last digit of your student id. number.]

Question 2 [35 marks]. Two dimensional systems

(a) For each of the following systems, find the fixed points and classify them, sketch the null-clines and the vector field, and suggest a plausible phase portrait.

(i)

$$\dot{x} = x + y, \quad \dot{y} = 1 - e^{-x}.$$

[6]

(ii)

$$\dot{x} = x^2 - y, \quad \dot{y} = x - y.$$

[6]

(b) Consider the system

$$\dot{x} = xy, \quad \dot{y} = -x^2.$$

(i) Show that the quantity $E(x,y) = x^2 + y^2$ is conserved over time. **[4]**

(ii) Show that the origin is not an isolated fixed point. **[4]**

(iii) Sketch the phase portrait. **[3]**

(c) A certain two dimensional system is known to have three fixed points, one saddle and two unstable nodes. Sketch a plausible phase portrait which has, as its only periodic orbits, the three fixed points described and a single stable limit cycle. **[6]**

(d) Find a dynamical system in polar coordinates in the form $\dot{r} = f(r,\theta)$, $\dot{\theta} = g(r,\theta)$, where f, g are suitably chosen functions, which exhibits a planar phase portrait with an unstable spiral focus at the origin surrounded by two circular limit cycles given by $r = 1$ (stable), and $r = 2$ (unstable), with anti-clockwise flow on the inner limit cycle, and clockwise flow on the outer limit cycle. **[6]**

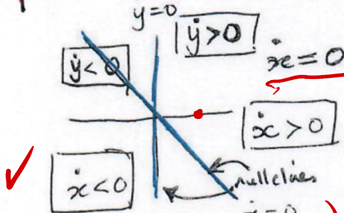
Q2 $\dot{x} = x+y, \quad \dot{y} = 1-e^{-x}$

Fixed points $x+y=0$, $1-e^{-x}=0$

$\Rightarrow y=-x$

$e^{x}=1 \Rightarrow x=0$

$\therefore x=0, y=0$ is unique fixed point.



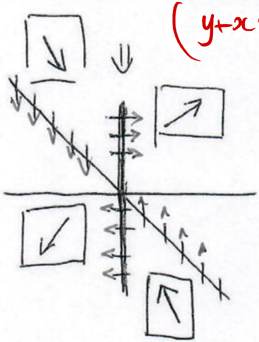
$\dot x = 0 : y = -x$

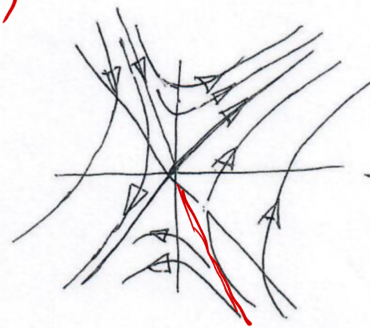
$(y+x=0)$

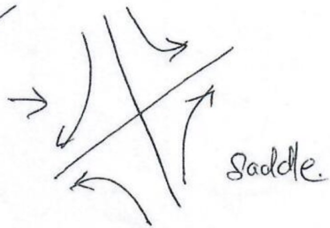
$\dot y = 0 \quad x \equiv 0$

$x>0$

$\dot y > 0$







general direction of flow   phase portrait → Saddle

$$L = \begin{bmatrix} 1 & 1 \\ e^{-x} & 0 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$\lambda_1 = \frac{1}{2}(1+\sqrt{5})$ $\quad v_1 = \left(\frac{1}{2}(1+\sqrt{5}), 1\right)$

$\lambda_2 = \frac{1}{2}(1-\sqrt{5})$ $\quad v_2 = \left(\frac{1}{2}(1-\sqrt{5}), 1\right)$

Linear analysis of fixed points

$\dot{x} = x^2 - y$

$\dot{y} = x - y$

$x^2 = x$

$x = y$

$y = x$; $x^2 - y = 0$ implies

$x^2 - x = 0 \Rightarrow x = 0, 1$

$y = 0, 1$

$x = 1, 0$

$y = 1, 0$

$\therefore$ Fixed points $(0,0)$ & $(1,1)$

$(0,0)$ $(1,1)$

Linearisation matrix $L = \begin{bmatrix} 2x & -1 \\ 1 & -1 \end{bmatrix}$

(Jacobian)

$(x,y) = (0,0)$: $L = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

Eigenvalues $(\lambda - 0)(\lambda+1) + 1 = 0 \Rightarrow \lambda^2 + \lambda + 1 = 0$

$\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ stable spiral

$(x,y)=(1,1)$:

$$L=\begin{bmatrix}2 & -1\\ 1 & -1\end{bmatrix}$$

(Jacobian)

Eigenvalues $(\lambda-2)(\lambda+1)+1=0 \Rightarrow \lambda^2-\lambda-1=0$

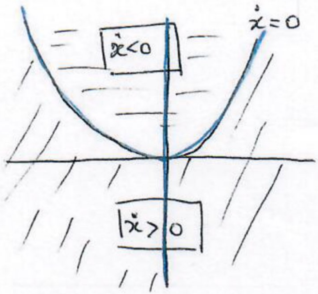
$\lambda=\frac{1}{2}\pm\frac{\sqrt{5}}{2}$ and $\frac{1+\sqrt{5}}{2}>0$, $\frac{1-\sqrt{5}}{2}<0$ $\therefore$ Saddle

$$\underline{v}_1=\left(\tfrac{1}{2}(3\pm\sqrt{5}),1\right)$$

$$\left(\tfrac{1}{2}(3+\sqrt{5}),1\right)=(2.6,1)$$

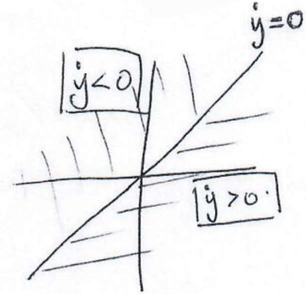
$$v_2=\left(\tfrac{1}{2}(3-\sqrt{5}),1\right)=(0.38,1)$$

2a (ii) $\dot{x} = x^2 - y, \quad \dot{y} = x - y.$

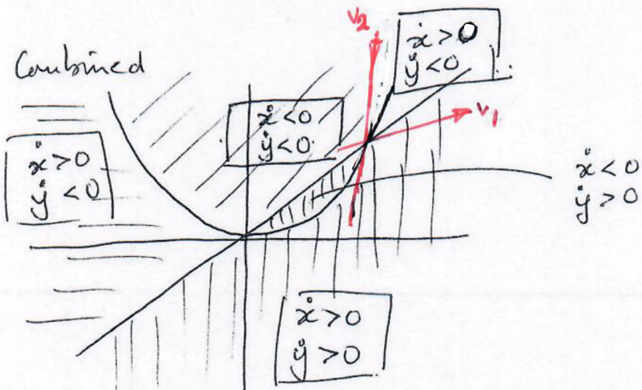


$\dot{x} = 0 \qquad y = x^2.$

$\dot{y} = 0 \qquad y = x$



Combined



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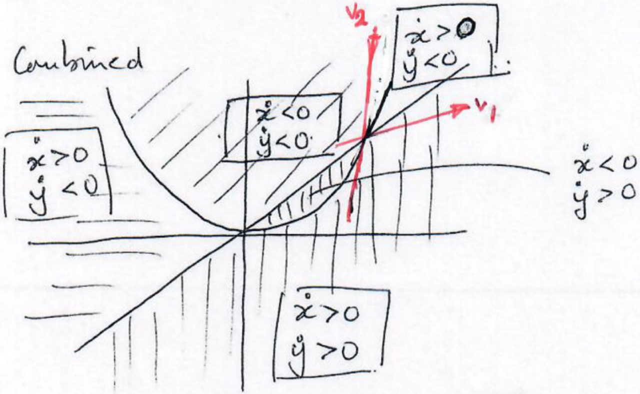
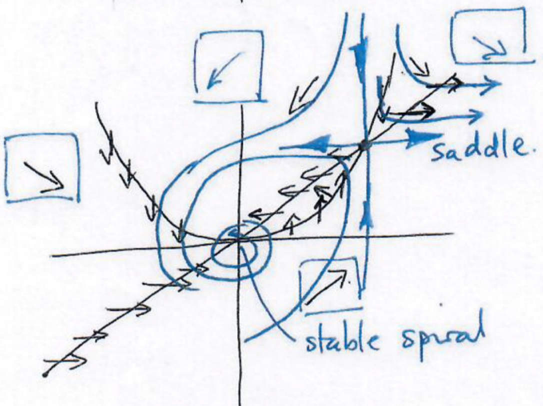
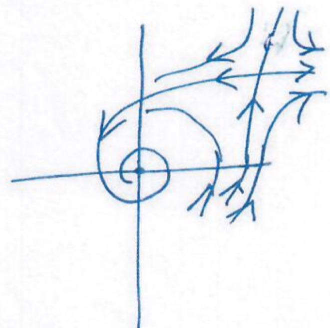
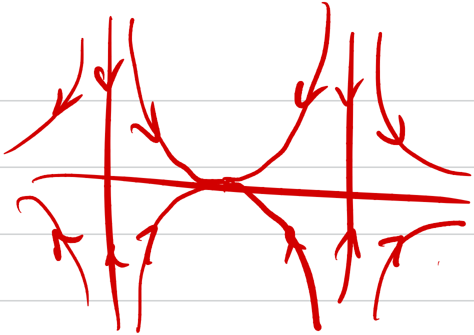
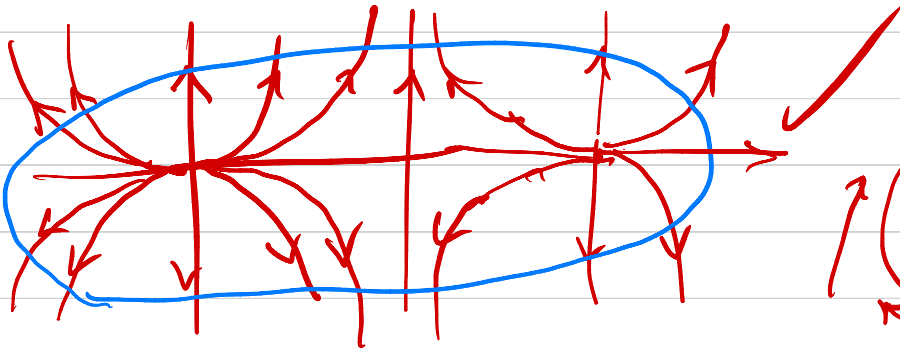

v_2
$\dot{x} > 0$
$\dot{y} < 0$
v_1
$\dot{x} < 0$
$\dot{y} < 0$
$\dot{x} > 0$
$\dot{y} < 0$
$\dot{x} < 0$
$\dot{y} > 0$
$\dot{x} > 0$
$\dot{y} > 0$

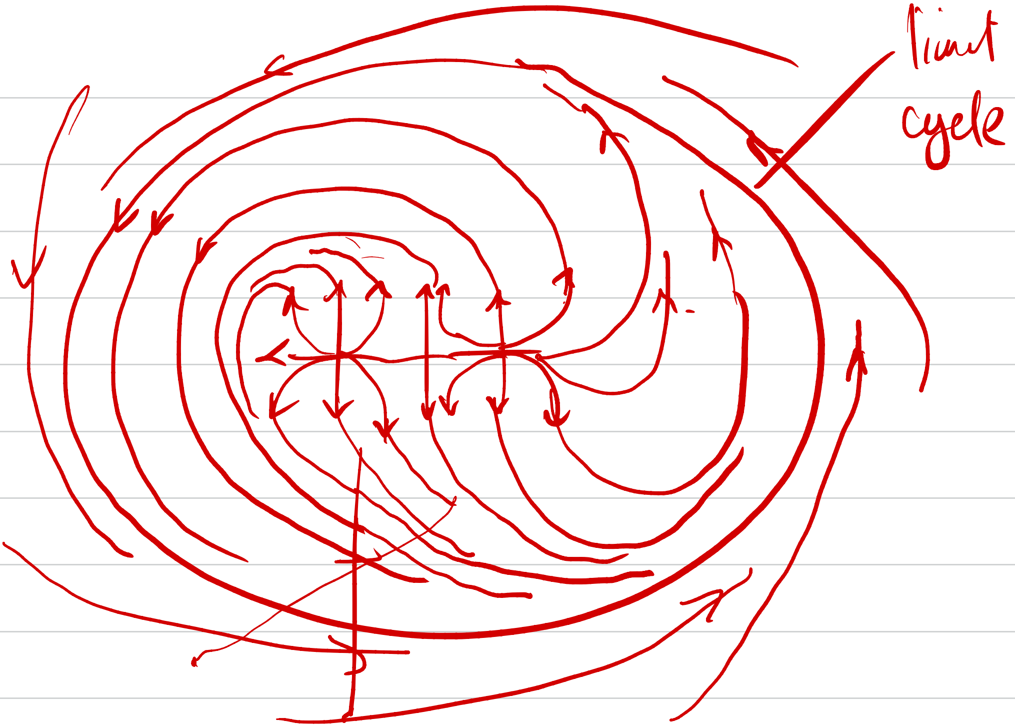

saddle.
stable spiral





from where




limit cycle.

WEEK 11

Ex 5.10 (Page 46)

Page 46 (i) $\dot{x} = \sin y$, $\dot{y} = x\cos y$

Gradient System cannot have periodic orbits period $T > 0$.

Is it a gradient system?

$\exists ?\, F$ s.t. $\dot{x} = -\frac{\partial F}{\partial x}$, $\dot{y} = -\frac{\partial F}{\partial y}$?

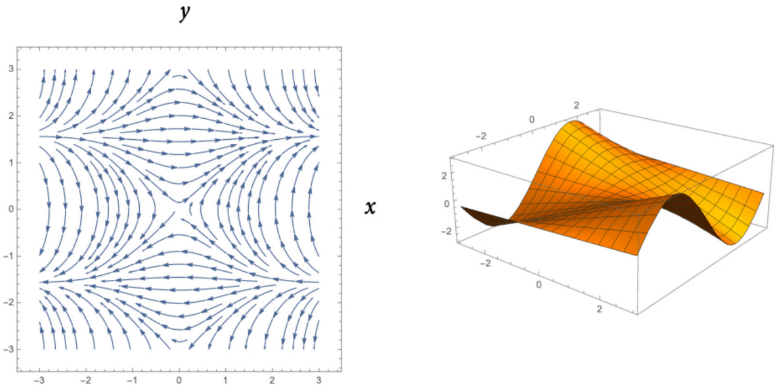
$F = F(x,y)$ $\quad \dot{x} = \sin y = -\frac{\partial F}{\partial x} \Rightarrow F \overset{?}{=} -x\sin y + \cancel{g(y)}$.

No periodic orbits $\quad$ but also $x\cos y \overset{\text{must}}{=} -\frac{\partial F}{\partial y}$ $\quad F = -x\sin y \quad \cancel{+ h(x)}$

(ii) Planar system obtained from ODE

$\ddot{x} + \dot{x}^3 + x = 0$ $\quad$ Gradient system.

WK 11. 1a (p46)



(a) (b)

Figure 26: *(a) The phase portrait of the system 5.27 , and (b) the corresponding surface plot, S,*
the potential $F(x, y) = -x\sin(y)$. Note that the vector field is not following the level curves of
but the perpendicular directions of steepest slope.

WK11.2

Example 5.10. *Consider the following systems:*

(i) The system

$$\dot{x} = \sin(y); \ \dot{y} = x\cos(y) \tag{5.27}$$

is a gradient system with potential function $F(x,y) = -x\sin(y)$. *So the system has no periodic orbits, see figure 26.*

(ii) The system obtained from the ODE $\ddot{x} + \dot{x}^3 + x = 0$ *has no periodic solutions. The corresponding system is* $\dot{x} = y$, $\dot{y} = -x - y^3$. *If the system is of gradient type, thenit could be concluded that there are no periodic orbits. This is not a gradient system: if a potential* $F(x,y)$ *existed, then* $\frac{\partial F}{\partial x}(\mathbf{x}) = -y$ *and* $\frac{\partial F}{\partial y}(\mathbf{x}) = x + y^3$, *but then we would have*

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right), \tag{5.28}$$

which is not true for this system. The condition 5.28 is, in fact, a necessary and sufficient condition for the existence of the potential function F. Hence, we need to show the existence of non-periodic solutions in a different way. Note that the linearised system at the origin is $\dot{x} = y$; $\dot{y} = -x$, *which has energy* $E(x,y) = \frac{1}{2}(x^2 + y^2)$ *which invites the possibility of an investigation using polar coordinates. Calculating* $\frac{dE}{dt}$, *we obtain* $\frac{dE}{dt} = x\dot{x} + y\dot{y} = -y^4 \leq 0$. *It follows that* $\Delta E = \int_0^T \dot{E}dt = 0$ *only if* $y(t) \equiv 0$ *along an orbit. But,*

$$y(t) \equiv 0 \implies \dot{x}(t) \equiv 0 \text{ and } \dot{y}(t) \equiv 0,$$

which means that the orbit is a fixed point and is not a period orbit.

$\ddot{x} + \dot{x}^3 + x = 0$ has no periodic orbits

$\boxed{\dot{x} = y}$, $\boxed{\dot{y} = \ddot{x} = -\dot{x}^3 - x = -y^3 - x}$

Not a gradient system.

$$y = -\frac{\partial F}{\partial x}, \quad -y^3 - x = -\frac{\partial F}{\partial y} \qquad F?$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) \neq \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

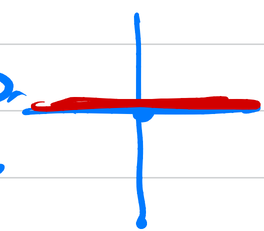
$= 1$ $\qquad$ $= -1$

Not gradient

For a closed orbit

$E = x^2 + y^2.$ $\qquad \frac{dE}{dt} = -y^4 \leq 0.$ $\qquad \frac{dE}{dt} \equiv 0$

$y \equiv 0$



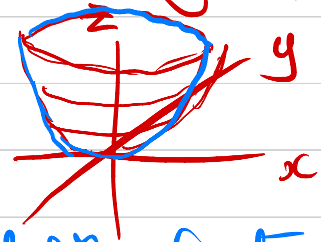
WK11.4

Liapounov Functions

Generalisation of the use of polar coordinates

$E(x,y) = x^2 + y^2.$

$z = x^2 + y^2$



positive defn function.

Defn 5.4 (p44) Let $U \subseteq \mathbb{R}^2$ be open set cont $\underline{0} \in \mathbb{R}^2$. Then $L : U \to \mathbb{R}$ is positive definite on U if $L(0,0) = 0$

$L(x,y) > 0 \quad \forall (x,y) \in U \setminus \{\underline{0}\}.$

L is s.t.b negative definite (ND) if $-L$ is PD

<u>Lemma</u> The quadratic function $L(x,y) = ax^2 + bxy + cy^2$
$a,b,c \in \mathbb{R}$ is PD iff $a > 0$ and $b^2 - 4ac < 0$

<u>Proof</u> Suppose L is PD $\therefore$ $L(1,0) = \underline{a > 0}$

Also $L(x,y) = a\left(x + \frac{b}{2a}y\right)^2 + \left(c - \frac{b^2}{4a}\right)y^2$

$ax^2 + bxy + \frac{a \cdot b^2}{4a^2}y^2$

Choose $x = -\frac{b}{2a}y$ $\quad L(x,y) = \left(c - \frac{b^2}{4a}\right)y^2$

$4ac - b^2 > 0$
$b^2 < 4ac$

$\therefore L(x,y) > 0 \Rightarrow c - b^2/4a > 0.$

WK11,6

Def^n 5.6, 5.7: PD L is Liapounov Function of $\underline{\dot{x}} = f(\underline{x})$ if $\frac{dL}{dt}(\underline{x}(t)) \leq 0$

PD L is a strict Liapounov Function (LP) of $\underline{\dot{x}} = f(\underline{x})$ if $\frac{dL}{dt}(\underline{x}(t)) < 0$.

Thm 5.5. $\underline{\dot{x}} = f(\underline{x})$ has a fixed pt at $\underline{x}^* = 0$.

The origin x^* is stable (Liapounov stable) if a L.F. exists (Def^n 5.6)

" is asymptotically stable if a STRICT LF exists.

The reverse is also true if $a>0$ and $b^2-4ac<0$

then $L(x,y) = a\left(x+\frac{by}{2a}\right)^2 + \left(c-\frac{b^2}{4a}\right)y^2$

with both coefficients non-zero! $\therefore L(x,y) \geq 0$

$\therefore L(x,y)=0 \Rightarrow y=0$ & $x+\frac{b}{2a}y=0 \Rightarrow$ $x=0$, $y=0$

$\therefore L(x,y)=0$ has the unique solution $(x,y)=(0,0)$.

Very simple examples

$E(x,y) = x^2 + y^2$, $\dot{x} = -x(x^2+y^2)$

$\dot{y} = -y(x^2+y^2)$

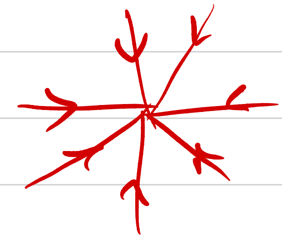
$\therefore$ E is a strict L.F.

$r\dot{r} = -(x^2+y^2)^2 = -r^4$

$\dot{r} = -r^3.$

$\dot{x} = y$, $\dot{y} = -x$

E is L.F.



Example 5.11

$\dot{x} = -x + 4y$, $\dot{y} = -x - y^3.$

$L(x,y) = x^2 + ay^2$, $a > 0$ LPD

$\dot{L}(x,y) = 2x\dot{x} + 2ay\dot{y} = 2x(-x+4y) + 2ay(-x-y^3).$

WK1

$= -2x^2 + 2(4-a)xy - 2ay^4$, $\underline{a > 0.}$

ND? $= -2x^2 - 8y^4$ $a = 4$

$\therefore \dot{L} < 0$ if $(x,y) \neq (0,0)$

$\therefore$ Asymptotic stability (globa<u>lly</u> a.s.)

Alt. $\underline{U = \mathbb{R}^2}$. [No restrictions on (x,y) to obtain the inequalities!].

$\dot{x} = 0, \; \dot{y} = 0$ WK 11.12

$3x\dot{x} + y\dot{y}$

(b) Consider the system of differential equations

$$\dot{x} = x(1 - 3x^2 - y^2) - y(1 + x), \; \dot{y} = y(1 - 3x^2 - y^2) + 3x(1 + x). \tag{4}$$

(i) Compute the fixed points of the system (4). For each fixed point determine the stability using linear stability analysis. [8]

(ii) Consider the quantity $L = (1 - 3x^2 - y^2)^2$. Show that $\frac{dL}{dt} \leq 0$. When does $\frac{dL}{dt} = 0$? [6]

(iii) Using the results of part (b)(ii), or otherwise, show that the system (4) has a unique limit cycle. Is the limit cycle stable or unstable? *Give reasons for your answer.* [6]

(iv) Using the results of part (b)(i-iii), or otherwise, sketch the phase portrait of the system (4). [4]

$\dot{x} = x(1-3x^2-y^2) - y(1+x)$, $\dot{y} = y(1-3x^2-y^2) + 3x(1+x)$

FPs $\dot{x}=0$, $\dot{y}=0$ $\quad 3x\dot{x} + y\dot{y} = 3x^2(1-3x^2-y^2) + y^2(1-3x^2-y^2)$
$\quad + \cancel{3xy}(1-3x^2-y^2) \cancel{-3xy(1+x)}$

$$= \boxed{(3x^2+y^2)(1-3x^2-y^2) = 0}$$

$\therefore \quad 3x^2+y^2=0 \Rightarrow x=y=0$ and $1-3x^2-y^3=0$

But $\dot{x}=\dot{y}=0$ also implies $1-3x^2-y^2=0$ and $\therefore \begin{cases} -y(1+x) = 0 \\ x(1+x) = 0 \end{cases}$

But $x=-1$ does not satify $1-3x^2-y^2=0$ $\forall$ any y.

$\therefore$ only fixed pr $=0$,

Now consider $L = (1-3x^2-y^2)^2 = E^2$, say, if $E \stackrel{\text{def}}{=} 1-3x^2-y^2$

Then $\dot{L} = 2E\dot{E} = 2E(-6x\dot{x} - 2y\dot{y})$

$$= -2E\left(-6x^2E + \cancel{6xy(1+x)} - 2y^2E - \cancel{6xy(1+x)}\right)$$

$$= -4E^2(3x^2+y^2)$$

$\therefore \quad \dot{L} = -4E^2(3x^2+y^2) = -4E^2(1-E)$

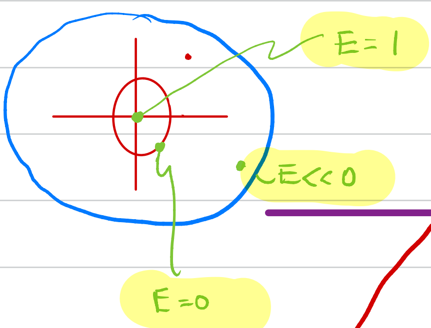
So $\dot{L} < 0$ except for when $E = 0$ or $(3x^2+y^2) = 0$

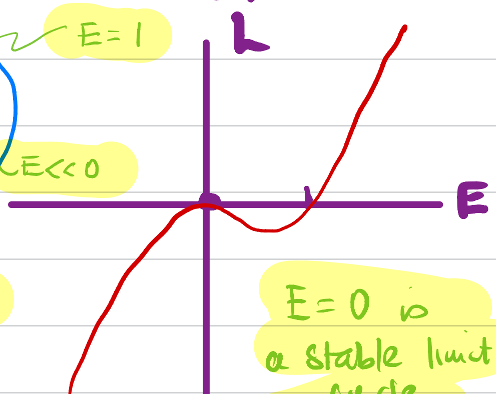
Note: $3x^2+y^2 = 0 \Leftrightarrow (x, y = 0)$

$\therefore \dot{L} < 0$ at pts other than $(x,y) = 0$.

and (x,y) satisfying $3x^2+y^2 = 1$ (i.e. $E = 0$)

Note $3x^2+y^2 = 1$ is an ellipse and $\begin{cases} \text{inside } 3x^2+y^2 < 1 \\ \text{outside } 3x^2+y^2 > 1 \end{cases}$





E = 0 is a stable limit cycle

$\therefore \dot{L} < 0$, for $0 < E < 1$

$\dot{L} > 0$, for $1 < E$ (imaginary (x,y))!

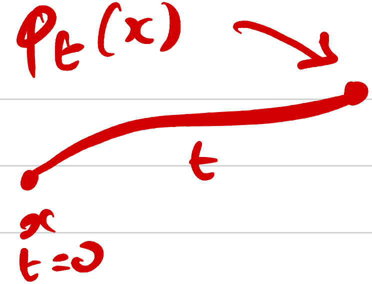
$E < 0$

$E = 1 \; : (x,y) = 0$ ↘ as $t \nearrow$

$E = 0 \; : (x,y) \in$ ellipse

$E < 0 \; : (x,y)$ ↗

WEEK 12



- Limit cycles
- α-limit sets
- ω-limit sets

Van der Pol Oscillator

$$\ddot{x} + \dot{x}\mu(x^2-1) + x = 0$$

or $\dot{x} = y$, $\dot{y} = -x - \mu y(x^2-1)$

Note $r\dot{r} = x\dot{x} + y\dot{y} = -\mu y^2(x^2-1)$. ?

$$r^2\dot{\theta} = -r^2 + \mu xy(x^2-1)$$

$$\dot{\theta} = -1 + \mu\cos\theta\sin\theta(x^2-1)$$?

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Example 5.12. ***Van der Pol Oscillator*** *This is a celebrated system discovered by the Dutch engineer in 1920. It modelled fluctuations and has been widely used in physics, engineering and biological modelling. Called the Van der Pol Oscillator, it has the form of a second order ODE in one real variable* x *as:*

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0,$$

where $\mu \in \mathbb{R}$ *is a parameter. The corresponding first order system in 2 variables,*

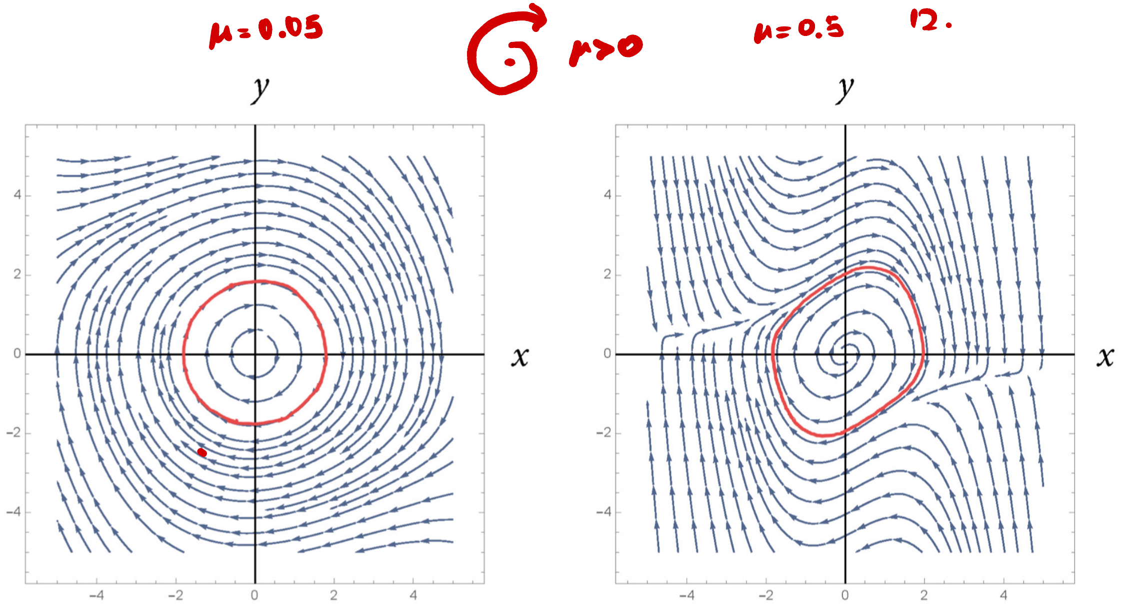
$$\dot{x} = y, \; \dot{y} = -x - \mu(x^2 - 1)y,$$

has a fixed point at the origin $(x, y) = (0, 0)$*. The Jacobian matrix is*

$$D\mathbf{f}(\mathbf{x}^*) = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix},$$

where $\mathbf{f}(\mathbf{x}) = (y, -x - \mu(x^2 - 1)y)$*, with eigenvalues* $\lambda = (\mu \pm \sqrt{(\mu^2 - 4)})/2$.

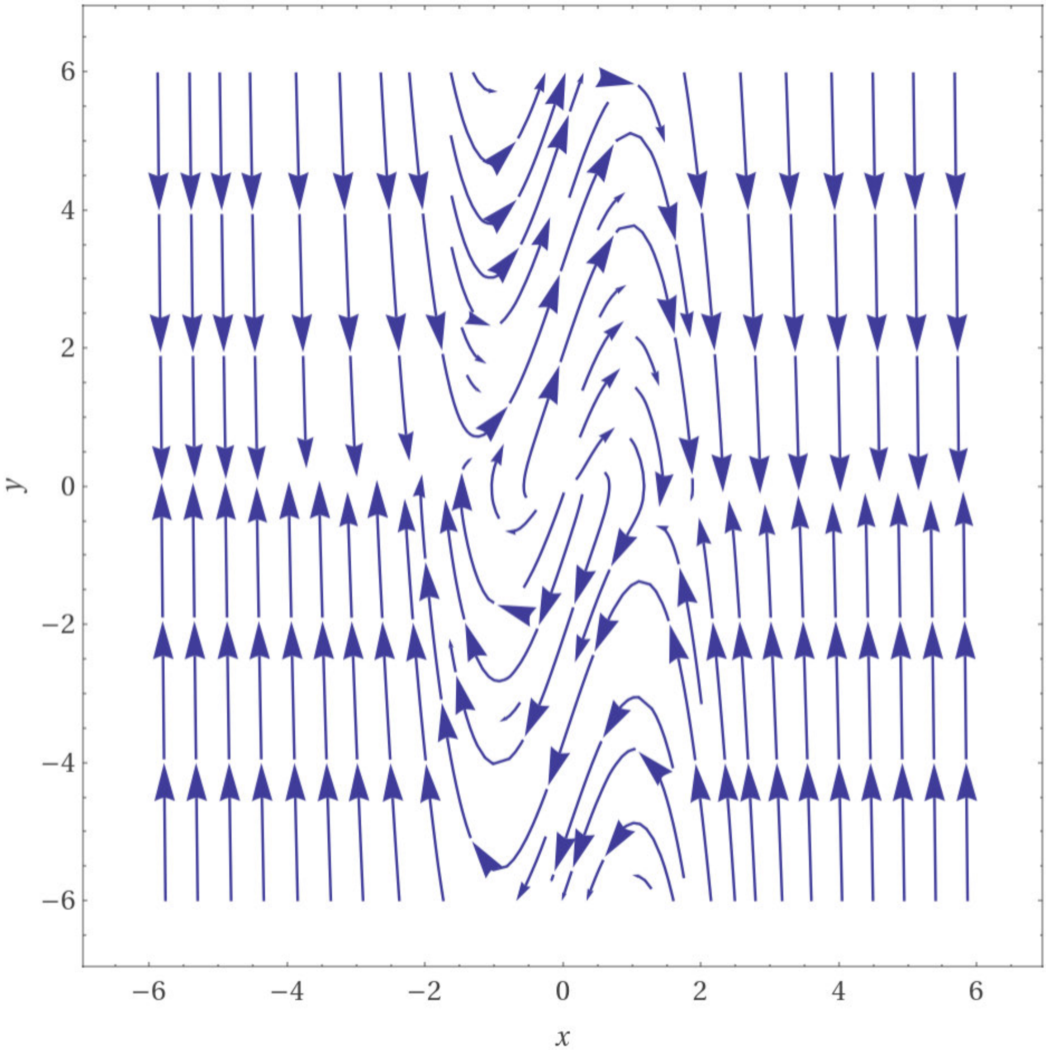
Therefore, by HGLT, we have spirals for $|\mu| < 2$ *which are unstable for* $\mu > 0$ *and stable for* $\mu < 0$*. It can also be shown that orbits with initial values at sufficiently large radial distance* r *spiral inwards which are then met by orbits spiralling out from the origin. The resulting collision of competing orbits is resolved by the existence of a stable limit cycle, see figure 28.*



(a) (b)

stream plot	$(y, -x - 3(-1 + x^2)\,y)$	$x = -6$ to 6 $y = -6$ to 6

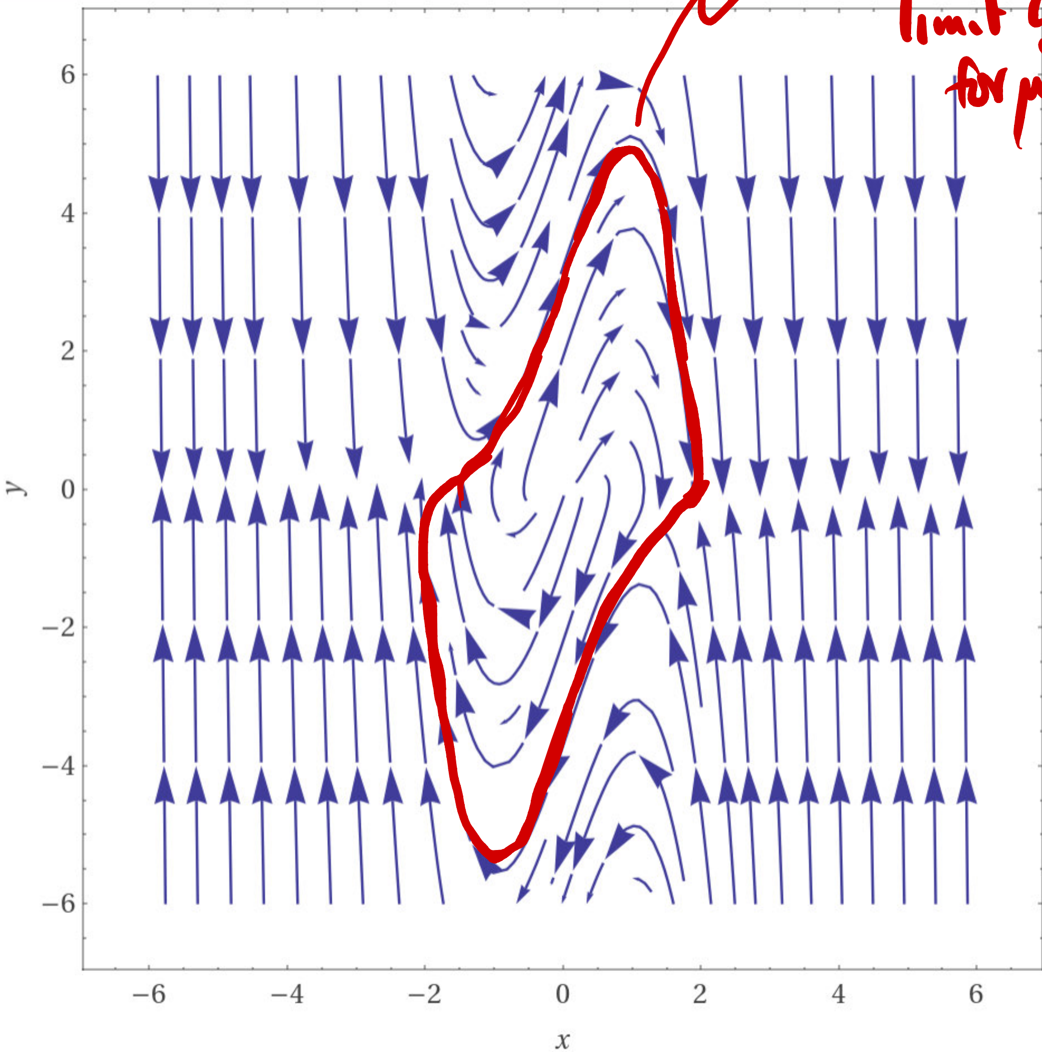
Plot



stream plot	$(y, -x-3(-1+x^2)\,y)$	$x = -6$ to 6 $y = -6$ to 6

Plot

approx position of VanderPol limit cycle for $\mu =$



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5.6.1 Poincaré-Bendixson Theorem

Definition 5.9. *For a planar system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, a point $\mathbf{y}$ is an element of the ω-limit set of the orbit $\mathbf{x} = \mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$ if there exists a sequence t_n with $\lim_{n\to\infty} t_n = \infty$ such that* $\lim \mathbf{x}(t_n) \to y$ *as $n \to \infty$. For the α-limit set, $\lim_{n\to\infty} t_n = \infty$ is replaced by $\lim_{n\to\infty} t_n = -\infty$.*

We consider the possible ω-limit sets $\omega = \omega(\mathbf{x}_0)$, where x_0 is the initial condition determining the orbit.

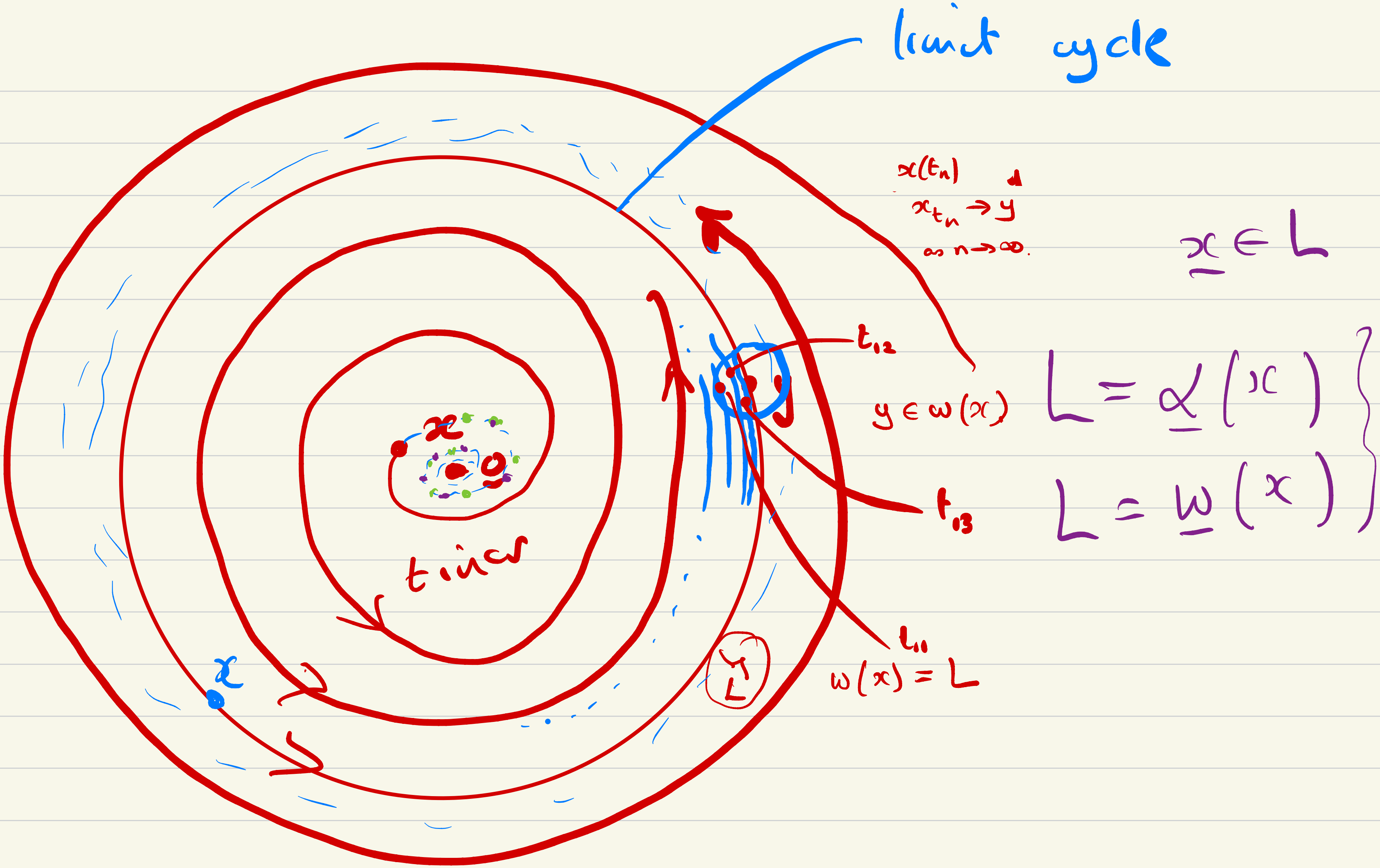
Example 5.14. *Consider the ω-limit of the following systems:*

(i) *$\dot{r} = -r$, $\dot{\theta} = 1$. All orbits converge to the origin which is a fixed point, i.e. $\dot{r} = -r$, $\dot{\theta} = 1$. All orbits converge to the origin which is a fixed point, i.e. $\omega(\mathbf{x}_0) = \{(0,0)\}$.*

(ii) *$\dot{r} = r(1-r)$, $\dot{\theta} = 1$. All orbits, apart from the origin, approach the unit circle, so there are two possible limit sets i.e. $\omega(\mathbf{0}) = \{(0,0)\}$ and $\omega(\mathbf{x}_0) = \{(x,y)|x^2+y^2=1\}$ for $\mathbf{x}_0 \neq \mathbf{0}$.*

(iii) *$\dot{r} = r(1-r)$, $\dot{\theta} = 1 - \cos(\theta) + (r-1)^2$. There are just two fixed points $\mathbf{x}_1^* = (0,0)$ and $\mathbf{x}_2^* = (1,0)$ All orbits on the unit circle Γ approach $\mathbf{x}_2^*$, so $\omega(\Gamma) = \{\mathbf{x}_2^*\}$. A more delicate investigation is needed to see that $\mathbf{x}_2^*$ is a saddle node fixed point with a stable eigen-direction on the x-axis, and a saddle node eigen-direction tangent to the circle. We are able to conclude that $\omega(\mathbf{x}_0) = \mathbf{x}_2^*$, $\forall \mathbf{x}_0 \in \mathbb{R}^2 \setminus \{\mathbf{x}_1^*)\}$, see figure 30.*

The following theorem describes the general case.



$\omega(x) = L$ for all $x \in \mathbb{R} \setminus \{0\}$

$\omega(0) = 0$

$\alpha(x) = 0$ for $x \in$ interior of

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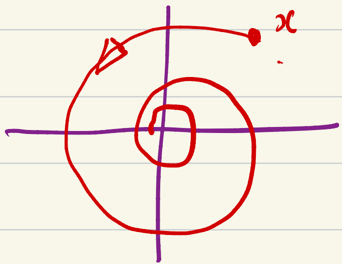
The following theorem describes the general case.

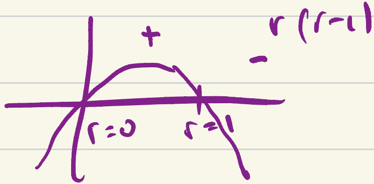
$\dot{r} = -r \quad , \quad \dot{\theta} = 1 \qquad \frac{dr}{d\tau} = \underline{\dot{r} = r \quad \dot{\theta} = -1} \; \frac{d\theta}{d\tau} \qquad$ Let $\tau = -t$

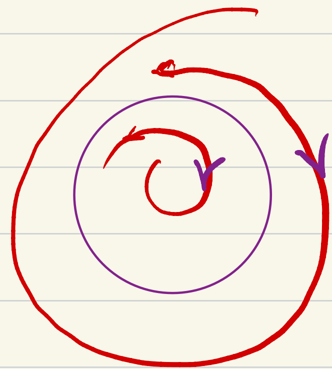


$\omega(x) = 0$

$\forall x \in \mathbb{R}^2$ incl 0.

$\underline{\dot{r} = r(1-r)} \qquad \dot{\theta} = 1$



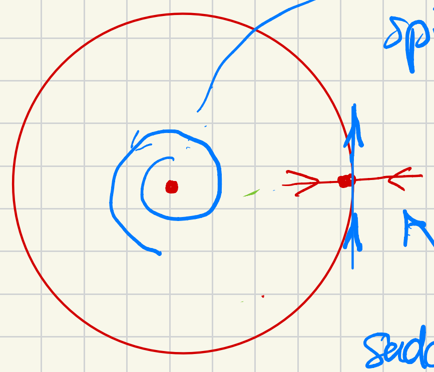


$$\dot{r} = r(1-r), \quad \dot{\theta} = 1 - \cos\theta + (r-1)^2$$

For fixed pts $\dot{r} = \dot{\theta} = 0$

$\therefore$ $r = 0$ or $r = 1$

$r = 0 \Rightarrow$ fixed pt at origin $= (0;0)$



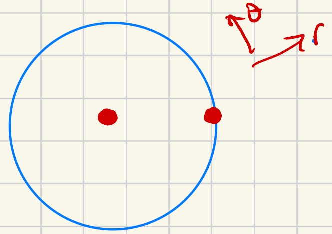
Note $r = 1$ is an invariant curve $r(t) \equiv 1$ satisfies $\dot{r} = r(1-r)$. (r_0, θ_0)

Dynamics on $r = 1$:

$$\dot{\theta} = 1 - \cos\theta + (r-1)^2 \Big|_{r=1} = 1 - \cos\theta$$

$\Longrightarrow$ $\dot{\theta} = 0$ when $\theta = 0$ only

$\therefore$ System has two fixed points $(r,\theta) = (0,0)$ and $(1,0)$



Local dynamics at

$(x,y) = (0,0)$, unstable Spiral $\quad \dot{r} \approx r$, $\dot{\theta} \approx 1 - \cos\theta + 1$

for $r \approx 0$

$(x,y) = (1,0)$ $\quad (r,\theta) = (1,0)$ also!

Local coords $\rho = r - 1$, $\theta = 0$

give $\dot{\rho} \approx -\rho$, $\dot{\theta} \approx \dfrac{\theta}{2}$

All orbits with initial pt $\underline{x}_0 \neq (0,0)$ have $\omega(\underline{x}_0) = (1,0) = P$

All orbits $\underline{x}_0 \neq (0,0)$ s.t. $r(\underline{x}_0) < 1$ have $\alpha(\underline{x}_0) \in (0,0)$

All orbits $\underline{x}_0 \in \Gamma$ have $\alpha(\underline{x}_0) = \omega(\underline{x}_0) = P$

All orbits $\underline{x}_0$ with $r(\underline{x}_0) > 1$ have $\omega(\underline{x}_0) = P$ and $\alpha(\underline{x}_0) = \infty$.

The following theorem describes the general case.

Theorem 5.6. ***(Poincaré-Bendixson)*** *Suppose that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a planar system with a finite number of fixed points. If the positive orbit $\mathbf{x}_0^+ = \{\mathbf{x}(t, \mathbf{x}_0);\ t \geq 0\}$, where $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$, is bounded, then one of the following is true.*

The ω-limit set $\omega(\mathbf{x}_0)$ is a

(a) *single point $\mathbf{x}^*$, which is a fixed point, and $\mathbf{x}(t, \mathbf{x}_0) \to \mathbf{x}^*$ as $t \to \infty$.*

(b) *periodic orbit Γ and either $\mathbf{x}_0^+ = \Gamma$, or $\mathbf{x}_0^+$ spirals towards Γ on one side of Γ.*

seen before

(c) *union of fixed points and orbits whose $\alpha-$ and $\omega-$ limit sets are fixed points. Such orbits are known as heteroclinic and homoclinic orbits, c.f. figure 29.*

new ish !

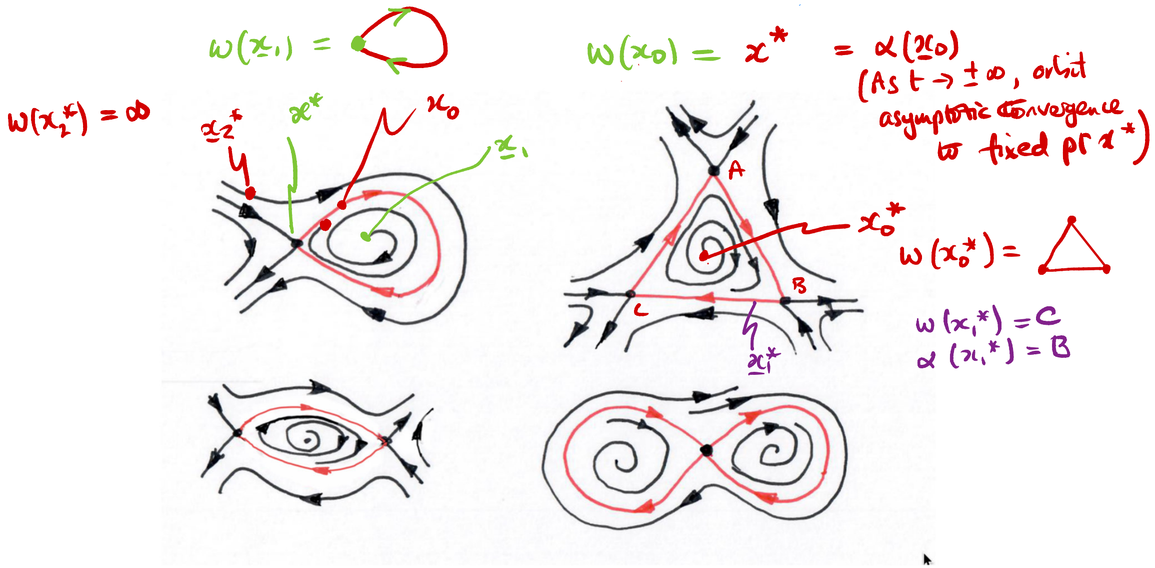
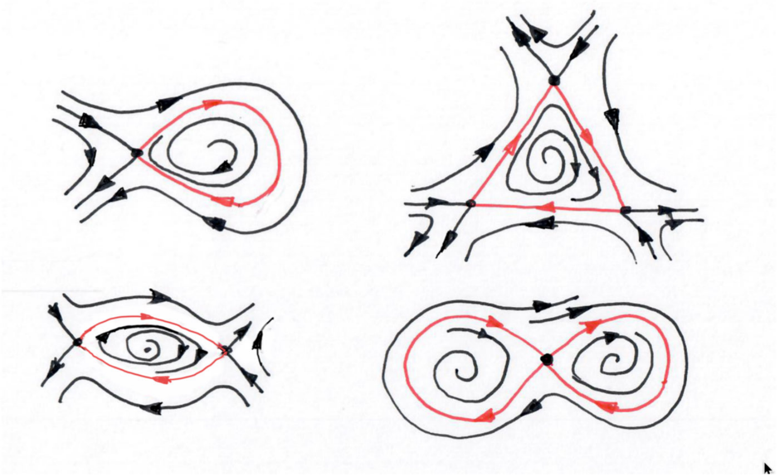


Figure 29: *Some examples of the α and ω-limit sets in phase portraits for case (c) of the Poincaré-Bendixson Theorem. The closed orbits, illustrated by the red curves, consist of a finite set of hetero-clinic or homoclinic orbits (these are orbits which connect saddle points).*



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