Selected solutions to PS 8.

1. (1) $\Delta(\ln r)$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\ln r)$$

$$= \left(\frac{\partial^2}{\partial r^2}\right)\ln r + \frac{1}{r}\frac{\partial}{\partial r}(\ln r) + 0$$

$$= \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + r \cdot \frac{1}{r} + 0$$

$$= \frac{-1}{r^2} + \frac{1}{r^2} = 0$$

So harmonic

1. (3) $\Delta\left(r^2 \cos 2\theta\right)$

$$= \frac{\partial^2}{\partial r^2}(r^2\cos 2\theta) + \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \cos 2\theta\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(r^2 \cos 2\theta)$$

$$= \frac{\partial}{\partial r}(2r\cos 2\theta) + \frac{1}{r}(2r\cos 2\theta) + \frac{1}{r^2}\left[r^2(-4\cos 2\theta)\right]$$

$$= 2\cdot\cos 2\theta + 2\cdot \cos 2\theta - 4\cos 2\theta$$

$$= 0$$

so also harmonic !

2. $\Delta v(x,y)$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[u(\lambda x, -\lambda y)]$$

$$= \frac{\partial^2}{\partial x^2}[u(\lambda x, -\lambda y)] + \frac{\partial^2}{\partial y^2}[u(\lambda x, -\lambda y)]$$

$$= \frac{\partial}{\partial x}\left[\lambda u_x(\lambda x, -\lambda y)\right] + \frac{\partial}{\partial y}\left[-\lambda u_y(\lambda x, -\lambda y)\right]$$

$$= \lambda^2 u_{xx}(\lambda x, -\lambda y) + (-\lambda)^2 u_{yy}[\lambda x, -\lambda y)$$

$$= \lambda^2 (u_{xx} + u_{yy})$$

$$= 0$$

so v is also harmonic.

also. for the domain of definitions.
we have u defined on $B_1(0)$,

so $(\lambda x, -\lambda y) \in B_1(0)$

Namely $\lambda^2 x^2 + \lambda^2 y^2 \leq 1$

$$x^2 + y^2 \leq \frac{1}{\lambda}$$

in other words

$$(x, y) \in B_{\frac{1}{\lambda}}(0)$$

v is defined on the disk $B_{\frac{1}{\lambda}}(0)$.

5 (1) Using the first mean value theorem

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} [3\sin\theta - 4\cos 4\theta + 1] d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 3\sin\theta \, d\theta - \frac{1}{2\pi}\int 4\cos 4\theta \, d\theta + \frac{1}{2\pi}\int_0^{2\pi} 1 \, d\theta$$

$$= 0 + 0 + 1$$

$$= 1$$

(2) By the maximum principle,

The max / min happens on the boundary.

Noticing $-1 \leq \cos\theta, \sin\theta \leq 1$,

we know on the boundary that

$$u(2,\theta) = 3\sin\theta - 4\cos 4\theta + 1$$

$$\leq 3 + 4 + 1$$

$$= 8$$

and $$u(2,\theta) = 3\sin\theta - 4\cos 4\theta + 1$$

$$\leq -3 - 4 - 1$$

$$= -6$$

So $-6 \leq u \leq 8$ on the whole Ω