PS 7 Q8:

The general solutions of Laplace equation in polar coordinate is

$$u(r,\theta) = C_0 + D_0 \ln r + \sum_{m=1}^{\infty} \left(C_m r^m + \frac{D_m}{r^m}\right)(A_m \cos m\theta + B_m \sin m\theta)$$

plug in the 2 boundary conditions, we get by "observation" that

the constant term has to agree, i.e.

$$\begin{cases} 1 = C_0 + D_0 \cdot \ln 1 = C_0 \\ 3 = C_0 + D_0 \cdot \ln e = C_0 + D_0 \end{cases}$$

so $\Rightarrow$ $C_0 = 1$, $D_0 = 2$

Next, because there are only $\sin\theta$ terms on the boundary data.

Using the fact that $\sin(m\theta)$ $\cos(n\theta)$ are independent, we must have.

$A_m = 0$ for all $m$

$B_m = 0$ for all $m$ except for $m=1$

Moreover, the ~~coefficients~~ ~~before~~ $\sin\theta$ need to match, so

$$\begin{cases} e = (C_1 \cdot 1 + D_1 \cdot 1) B_1 \\ 1 = (C_1 e + \frac{D_1}{e}) B_1 \end{cases}$$

This equation has more unknowns than restrictions, we can set $B_1 = 1$,

and get

$$\begin{cases} e = C_1 + D_1 \\ 1 = e \cdot C_1 + \frac{1}{e} \cdot D_1 \end{cases}$$

Solve it, we get $\begin{cases} C_1 = 0 \\ D_1 = e \end{cases}$

So the solution is

$u(r,\theta) = 1 + 2\ln r + \frac{e}{r}\sin\theta$