Week 8

What we'll cover (lecture 1)

A] Rank-nullity for matrices (examples)

B] Change of basis

A] From Week 6 2] Ex 2

$$L(A) = A - A^T \Rightarrow \text{nul}(L) = 3, \ \text{rank}(L) = 1$$

From Week 6 4] Ex 2

The matrix associated to L with respect to the basis B defined in 4] Ex 2 is

$$D = [L]_B^B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{REF}{\sim} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3 free variables $\Rightarrow$ $\text{nul}(D) = 3$ in agreement with the result above!

1 leading variable $\Rightarrow$ $\text{Rank}(D) = 1$

- Similarly from Week 6 2] Ex 1

$$D = \frac{d}{dx} \quad P_3 \to P_2$$ and

$\operatorname{rank}(D) = 3$, $\operatorname{nul}(D) = 1$

From Week 6 4] Ex 1

The matrix associated to D with respect to the standard basis B (defined in week 6) is

$$[D]_B^B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$\operatorname{rank}([D]_B^B) = 3$ (three non-trivial rows)

$\operatorname{nul}([D]_B^B) = 1$ (1 free variable)

in agreement with the result above

B] Change of Basis

<u>def</u>: Let B, B' be two basis for a vector space U of dimension n. The matrix

$$P_{B,B'} = [id]_{B'}^{B}$$ (id is the identity operator)

is the <u>transition matrix</u> from B to B'

We can construct this matrix by calculating the coordinates of the vectors in the (old) basis according to the (new) basis B'. These coordinates are the column vectors of $P_{B,B'}$

Simple sanity check

Lemma 1 if $B' = B$ then

It follows from the definition of P

$$P_{B,B} = I \quad \text{(the identity } n \times n \text{ matrix)}$$

Lemma 2 : $P_{B',B''} \cdot P_{B,B'} = P_{B,B''}$

Application of a corollary discussed on Week 6 (p.10)

Lemma 3 $\quad P_{B',B} \, P_{B,B'} = P_{B,B} = I \Rightarrow P_{B',B}^{-1} = P_{B,B'}$

(It follows from lemma 1,2).

Thus in general we have

$$\boxed{\underline{b}_i = \sum_{j=1}^{n} (P_{B,B'})_{ji} \, \underline{b}'_j}$$

where $B = \{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$

$B' = \{\underline{b}'_1, \underline{b}'_2, \dots, \underline{b}'_n\}$

The transition matrix is

$$P = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & & \vdots \\ \vdots & \vdots & & \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{pmatrix}$$

What we'll cover (lectures 2→3)

4] Change of basis examples

5] Change of basis, linear maps, matrices

6] Isomorphism

7] Eigenvectors and eigenvalues: definition

4] Ex 1. Let

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ and } B' = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$$

They are two basis for $\mathbb{R}^2$. Let us derive the the transition matrix $P_{B,B'}$ from B to B'.

By applying the general equation above, we have

This is for $P_{B,B'}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = P_{11} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + P_{21} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \begin{array}{l} P_{11} = -1 \\ P_{21} = \frac{1}{2} \end{array}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = P_{12} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + P_{22} \begin{pmatrix} 0 \\ 2 \end{pmatrix} \Rightarrow \begin{array}{l} P_{12} = 0 \\ P_{22} = \frac{1}{2} \end{array}$$

Thus we have

$$P_{B,B'} = \begin{pmatrix} -1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

In this case it was easier to calculate $P_{B',B}$. Then we have to solve the same problem, but with the roles of $\underline{b}_i$ and $\underline{b}'_i$ swapped

$$\underline{b}'_i = \sum (P_{B',B})_{ji} \, \underline{b}_i$$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = P_{11} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + P_{21} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} P_{11} = -1 \\ P_{21} = 1 \end{array}$$

$$\begin{pmatrix}0\\2\end{pmatrix} = P_{12}\begin{pmatrix}1\\0\end{pmatrix} + P_{22}\begin{pmatrix}0\\1\end{pmatrix} \qquad \begin{matrix}P_{12} = 0\\ P_{22} = 2\end{matrix}$$

$$\Rightarrow \quad P_{B',B} = \begin{pmatrix}-1 & 0\\ 1 & 2\end{pmatrix}$$

We can check lemma 3 above

$$P_{B,B'}\,P_{B',B} = \begin{pmatrix}-1 & 0\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}-1 & 0\\ 1 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Thus we have $P_{B,B'} = P_{B',B}^{-1}$

Let us check that the name "transition" matrix makes sense:

- the coordinates of $\underline{b}_1'$ in the basis B' are

$$[\underline{b}_1']_{B'} = \begin{pmatrix}1\\0\end{pmatrix} \qquad \text{(of course)}$$

- the coordinates of b_1' in the basis B ar

$$[\underline{b}_1']_B = \begin{pmatrix}-1\\1\end{pmatrix}$$

Now you can check that $P_{B',B}$ maps $\begin{pmatrix}1\\0\end{pmatrix}$ to $\begin{pmatrix}-1\\1\end{pmatrix}$ so it maps the coordinates according to B' to the coordinates according to B of the same vector

5] A similar property holds also for linear maps

Theorem: (change of basis formula):

Let B, B' be basis for a vector space U

" C, C' " " " " " " V

and $L: U \to V$ be a linear map. Then

$$[L]^{B'}_{C'} = [id]^{C}_{C'} [L]^{B}_{C} [id]^{B'}_{B}$$

It follows as an application of the proposition on pag. 9 of week 6 (see Th. 5.51 of the typewritten notes).

A particular, but important case is when $U=V$.

Consider a map $M: V \to V$ then

$$[M]^{C'}_{C'} = [id]^{C}_{C'} [M]^{C}_{C} [id]^{C'}_{C}$$

reads for the associated matrice

$$[M]^{C'}_{C'} = P_{C,C'} [M]^{C}_{C} P_{C',C} =$$

$$= P_{C,C'} [M]^{C}_{C} P^{-1}_{C,C'}$$

The matrices $[M]^{C'}_{C'}$ and $[M]^{C}_{C}$ above are said to be <u>similar</u> (and a transformation $M \to PMP^{-1}$ is called similarity transformation).

6] ISOMORPHISM

<u>Definition</u>: let V, W be vector spaces. A linear map $L: V \to W$ is an isomorphism if there is a linear map $M: W \to V$ such that

$$L \circ M = id \qquad M \circ L = id$$

Then the spaces V and W are said to be isomorphic (indicated by $V \cong W$).

Lemma: $L: V \to W$ is an isomophis iff

L is bijective

Sketch of a proof: if L is bijective, then there exists an inverse function M. Now you have just to prove that M is a <u>linear</u> map (by using that L is linear). See the typewritten notes. Then we have

Proposition: Let $L: V \to W$ be an isomorphism. and S is a set of linearly independent vectors in V. Then

1) $L(S)$ is a set of linearly independent vectors in W
2) if S spans V (i.e. it is a basis), then $L(S)$ spans W (" " ").

Consequences of the bijective property of L

<u>Corollary</u> If $V \cong W$ iff $\dim(V) = \dim(W)$

$\Rightarrow$ A consequence of the proposition before

$\Leftarrow$ Choose a basis $(\underline{v}_1, \dots, \underline{v}_n)$ in V and a basis $(\underline{w}_1, \dots, \underline{w}_n)$ in W. The unique linear map $L(\underline{v}_i) = \underline{w}_i$ is an isomorphism

Corollary: $\dim(V) = n \Rightarrow V \cong \mathbb{R}^n$

Thus we can always represent any finite dim. vector space as $\mathbb{R}^n$ and the linear maps between vector spaces as matrices! This is what we did when we introduced coordinates

7] Eigenvector / eigenvalues

Let V be a vector space of dimension n and $L: V \to V$ be a linear map. An eigenvector of L is a non-zero vector $\underline{v} \in V$ which satisfies

$$L\underline{v} = \lambda \underline{v} \quad \text{for some } \lambda \in \mathbb{R}$$

(the same is true with $\lambda \in \mathbb{C}$ for complex vect. spac.)

λ is called eigenvalue of A and $\underline{v}$ an eigenvector corresponding to λ.

Since V is isomorphic to $\mathbb{R}^n$ and L can be represented by a matrix A once we choose a basis, we have that $\underline{x} \in \mathbb{R}^n$ $([\underline{v}]_B = \underline{x})$ is an eigenvector of $A = [L]_B^B$ if

$$A\underline{x} = \lambda \underline{x} \quad (\underline{x} \neq \underline{0} \text{ since } \underline{v} \neq \underline{0}) \quad \lambda \in \mathbb{R}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$. Consider $\underline{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\underline{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$A\underline{x}_1 = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_1 \underline{x}_1$$

So $\begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector and $\lambda_1 = 3$ is the corresponding eigenvalue

$$\begin{pmatrix}1 & 2\\-1 & 4\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4\\2\end{pmatrix} = 2\begin{pmatrix}2\\1\end{pmatrix} = \lambda_2 \underline{x}_2$$

So this is another eigenvector with eigenvalue $\lambda_2 = 2$.

Notice that the eigenvector $\underline{x}_1$ corresponding to λ_1 is non-unique. We can choose $\underline{x}_1^{new} = \begin{pmatrix}2\\2\end{pmatrix}$ and we would get again a solution to the eigenvector/eigenvalue equation with $\lambda_1 = 3$

$$A\underline{x}_1^{new} = \begin{pmatrix}1 & 2\\-1 & 4\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}6\\6\end{pmatrix} = 3\begin{pmatrix}2\\2\end{pmatrix} = \lambda_1 \underline{x}_1^{new}$$

Clearly any $c\,\underline{x}_1$ (with $c \in \mathbb{R}$ and $c \neq 0$) is a non-trivial solution with $\lambda_1 = 3$ (and similarly for $c\underline{x}_2$ and λ_2).

How to find systematically eigenvectors/eigenvalues? When/why are they useful? Stay tuned ...