Selected solutions for problem set 7.

1. First separate variables. Look for solutions of the form

$$u(x,y) = X(x)\,Y(y)$$

Get $X''Y + XY'' = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = K$$

Get eigenvalue problems

$$(*1)\quad \begin{cases} Y'' + KY = 0 \\ Y(0) = 0 \\ Y(b) = 0 \end{cases}$$

and

$$(*2)\quad \begin{cases} X'' - KX = 0. \\ X(0) = g_1(y) \\ \cancel{X(b) = 0}\ X(a) = 0 \end{cases}$$

claim: $K \geq 0$

proof of claim: $\int_0^b Y \cdot Y'' + KY^2 = 0$ Multiply equation by Y and integrate

$$YY'\Big|_0^b - \int_0^b (Y')^2 + K\int_0^b Y^2 = 0$$

~~so $K > 0$.~~ $K\int_0^b Y^2 = \int_0^b (Y')^2$

So $K \geq 0$.

Thus the general solutions are

$$Y(y) = A\cos(\sqrt{K}y) + B\sin(\sqrt{K}y)$$

~~$A\cos m$~~

~~$A\cos(my) + B\sin(my)$, where write $K = m^2 \geq 0$~~

Using $Y(0)=0$, we get.

$$A = Y(0) = 0$$

so $B \neq 0$

Using $Y(b)=0$, we get

$$B \sin(\sqrt{k}\, b) = Y(b) = 0$$

Thus $\sqrt{k}\, b = n\pi, \quad n = 1, 2, \cdots$

Next solve $X(x)$, get

$$X'' - \frac{n^2\pi^2}{b^2} X = 0.$$

So $\quad X_n(x) = \tilde{C_n} \cosh\left(\frac{n\pi(x-a)}{b}\right) + \tilde{D_n} \sinh\left(\frac{n\pi(x-a)}{b}\right)$

Using that ~~$X(b)=0$~~, $X(a)=0$

we get $\tilde{C_n} = 0$ and

$$X_n(x) = \tilde{D_n} \sinh\left(\frac{n\pi(x-a)}{b}\right)$$

So $U_n(x,y) = X_n(x)\, Y_n(y) = \tilde{D_n} B_n \sin\frac{n\pi y}{b} \sinh\left[\frac{n\pi(x-a)}{b}\right]$

and the general solutions are

$$U(x,y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi(x-a)}{b}$$

where $D_n = \tilde{D_n} B_n$

To fix D_n, we use $U(0,y) = g_1(y)$, i.e.

$$\sum_{n=1}^{\infty} - D_n \sin\frac{n\pi y}{b} \sinh\frac{n\pi a}{b} = g_1(y)$$

So $\quad D_n = - \dfrac{\int_0^b g_1(y) \sin\frac{n\pi y}{b}\, dy}{\frac{b}{2} \sinh\frac{n\pi a}{b}}$

And the solution to this Dirichlet problem is ③

$$u(x,y) = -\sum_{n=1}^{\infty} \frac{\int_0^b g_1(y)\sin\frac{n\pi y}{b}\,dy}{\frac{b}{2}\sinh\frac{n\pi a}{b}} \sin\frac{n\pi y}{b}\sinh\frac{n\pi(x-a)}{b}.$$

3. we use the principle of superposition and consider the following 2 problems.

$$(*3)\begin{cases} \Delta v = 0 \\ v(x,0) = x \\ v(x,1) = 0 \\ v(0,y) = 0 \\ v_x(1,y) = 0 \end{cases} \qquad (*4)\begin{cases} \Delta w = 0 \\ w(x,0) = 0 \\ w(x,1) = 0 \\ w(0,y) = 0 \\ w_x(1,y) = y^2 \end{cases}$$

First solve $(*_3)$ by separation of variables.

suppose $v(x,y) = $ ~~$v(x)v(y)$~~ $X(x)Y(y)$

get $$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

and $\begin{cases} X'' - kX = 0 \\ X(0) = 0 \\ X'(1) = 0 \end{cases} \qquad \begin{cases} Y'' + kY = 0 \\ Y(0) = x \\ Y(1) = 0. \end{cases}$

claim: $k \leq 0$.

proof of claim: $\int_0^1 XX'' - \int_0^1 kX^2 = 0$ ← Multiply equation by X and integrate.

$$XX'\big|_0^1 - \int_0^1 (X')^2 - k\int X^2 = 0$$

$= 0$ by boundary condition. $\quad k\int_0^1 X^2 = -\int (X')^2$

So $k \leq 0$

So $X(x) = A\cos(\sqrt{-k}\,x) + B\sin(\sqrt{-k}\,x)$.

using $X(0)=0$, we get $A=0$

and so $B \neq 0$.

using $X'(1)=0$, we get

$$X'(1) = B\sqrt{-k}\cos(\sqrt{-k}x)\Big|_{x=1} = 0.$$

$$\sqrt{-k} = \frac{\pi}{2} + n\pi \quad \text{for } n=1,2,\cdots$$

$$k = -\left(\frac{1}{2}+n\right)^2\pi^2$$

Next solve $Y(y)$, get

$$Y'' - \left(\frac{1}{2}+n\right)^2\pi^2 Y = 0$$

$$\cancel{Y(y) = \tilde{C}_n \cos\left[\left(\frac{\pi}{2}+n\pi\right)y\right] + \tilde{D}_n \sin}$$

$$Y(y) = \tilde{C}_n \cosh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right] + \tilde{D}_n \sinh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right]$$

Using that $Y(1)=0$, get

$$\tilde{C}_n = 0,$$

So $V_n(x,y) = X_n(x)\, Y_n(y)$

$$= B_n \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right] \cdot \tilde{D}_n \sinh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right]$$

$$= D_n \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right] \sinh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right]$$

where $D_n = B_n \cdot \tilde{D}_n$

and the General solutions are

$$V(x,y) = \sum_{n=1}^{\infty} D_n \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right] \sinh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right]$$

To fix D_n, we use $V(x,0) = x$, i.e.

$$x = \sum_{n=1}^{\infty} -D_n \sin\left(\left(\frac{\pi}{2}+n\pi\right)x\right) \cdot \sinh\left(\frac{\pi}{2}+n\pi\right)$$

so $$D_n = -\frac{\int_0^1 x \cdot \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right]dx}{\frac{1}{2}\cdot \sin\left(\frac{\pi}{2}+n\pi\right)}$$

$$= -2\int_0^1 x\cdot \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right]dx$$

and the solution is

$$V(x,y) = -\sum_{n=1}^{\infty} 2\left\{\int_0^1 x\cdot \sin\left[\left(\frac{\pi}{2}+n\pi\right)x\right]dx\right\} \sin\left(\frac{\pi}{2}+n\pi\right)x \cdot \sinh\left[\left(\frac{\pi}{2}+n\pi\right)(y-1)\right]$$

~~Step~~ After getting V, we solve (*4) to get W.

Similarly, suppose $W(x,y) = X(x)\,Y(y)$

get $$\frac{X''}{X} = -\frac{Y''}{Y} = k.$$

and $\begin{cases} X''-kX=0 \\ X(0)=0 \\ X'(1)=y^e \end{cases}$, $\begin{cases} Y''+kY=0 \\ Y(0)=0 \\ Y(1)=0. \end{cases}$

First solve Y,

$\boxed{\text{claim: } k \geq 0. \quad (\text{The proof is the same as in lecture})}$

get $Y(y) = A\cos(\sqrt{k}y) + B\sin(\sqrt{k}y)$

using $Y(0)=0$, get $A=0$, $B\neq 0$

using $Y(1)=0$, get $\sqrt{k}=n\pi$,

so $k = n^2\pi^2$, $n=0,1,\cdots$

Next solve X,

$$X''-n^2\pi^2X=0$$

so $$X_n(x) = \tilde{C_n}\cosh(n\pi x) + \tilde{D_n}\sinh(n\pi x)$$

using that $X(0)=0$ get $\bar{C}_n = 0$.

So ~~solution for W is~~
general solution for W is

$$W(x,y) = \sum_{n=0}^{\infty} X_n(x)\, Y_n(y)$$

$$= \sum_{n=0}^{\infty} D_n \sin(n\pi y) \sinh(n\pi x)$$

where $D_n = B_n \cdot \bar{D}_n$.

To fix D_n, we use

$$y^2 = W_x(1,y) = \sum_{n=0}^{\infty} \cancel{D_n (n\pi) \cos(n\pi y)}\ \text{si}$$
$$D_n \cdot (n\pi) \sin(n\pi y) \cdot \cosh(n\pi x)\big|_{x=1}.$$

namely

$$y^2 = \sum_{n=0}^{\infty} D_n \cdot (n\pi) \cdot \frac{e^{n\pi} + e^{-n\pi}}{2} \cdot \sin(n\pi y)$$

thus
$$D_n = \frac{\int_0^1 y^2 \cdot \sin(n\pi y)\, dy}{\frac{1}{2} \cdot (n\pi) \cdot \frac{e^{n\pi} + e^{-n\pi}}{2}}$$

and
$$W(x,y) = \sum_{n=0}^{\infty} \frac{4}{(n\pi) \cdot (e^{n\pi} + e^{-n\pi})} \cdot \left[\int y^2 \sin(n\pi y)\, dy\right] \sin(n\pi y) \sinh(n\pi x)$$

The solution to the original problem is

$$u(x,y) = v(x,y) + w(x,y)$$

$$= -\sum_{n=1}^{\infty} 2 \cdot \left\{\int_0^1 x \cdot \sin\left[\left(\frac{\pi}{2} + n\pi\right)x\right] dx\right\} \cdot \sin\left[\left(\frac{\pi}{2} + n\pi\right)x\right] \sinh\left[\left(\frac{\pi}{2} + n\pi\right)(y-1)\right]$$

$$+ \sum_{n=0}^{\infty} \frac{4}{(n\pi)(e^{n\pi} + e^{-n\pi})} \cdot \left[\int y^2 \sin(n\pi y)\, dy\right] \sin(n\pi y) \sinh(n\pi x).$$

5. $$u(r,\theta) = \frac{(r^2 - r_*^2)}{2\pi} \int_0^{2\pi} \frac{f(\theta')\,d\theta'}{r^2 - 2rr_*\cos(\theta-\theta') + r_*^2}$$

$$= \frac{(r^2 - r_*^2)u_1}{2\pi} \int_0^{\pi} \frac{d\theta'}{r^2 + r_*^2 - 2rr_*\cos(\theta-\theta')}$$

$$+ \frac{(r^2 - r_*^2)u_2}{2\pi} \int_{\pi}^{2\pi} \frac{d\theta'}{r^2 + r_*^2 - 2rr_*\cos(\theta-\theta')}$$

Using the hint, we get

$$u(r,\theta) = -\frac{(r^2 - r_*^2)u_1}{2\pi} \cdot \frac{2}{r_*^2 - r^2} \arctan\left[\frac{r_*+r}{r_*-r}\tan\left(\frac{\theta-\theta'}{2}\right)\right]\Bigg|_0^{\pi}$$

$$- \frac{(r^2 - r_*^2)u_2}{2\pi} \cdot \frac{2}{r_*^2 - r^2} \arctan\left(\frac{r_*+r}{r_*-r}\tan\left(\frac{\theta-\theta'}{2}\right)\right)\Bigg|_{\pi}^{2\pi}$$

$$= -\frac{u_1}{\pi}\arctan\left(\frac{r_*+r}{r_*-r}\cot\frac{\theta}{2}\right) + \frac{u_1}{\pi}\arctan\left(\frac{r_*+r}{r_*-r}\tan\frac{\theta}{2}\right)$$

$$+ \frac{u_2}{\pi}\arctan\left(\frac{r_*+r}{r_*-r}\tan\frac{\theta}{2}\right) + \frac{u_2}{\pi}\arctan\left(\frac{r_*+r}{r_*-r}\cot\frac{\theta}{2}\right)$$

6. using the general formula in polar coordinates

$$u(r,\theta) = (C_0 + D_0 \ln r) + \sum_{m=1}^{\infty} \left(C_m r^m + \frac{D_m}{r^m}\right)(\cos m\theta + \sin m\theta).$$

since
$$u(r_1,\theta) = u_1$$
$$u(r_2,\theta) = u_2$$
are independent of θ

we have all $C_m, D_m = 0$ for $m \geq 1$.

and
$$C_0 + D_0 \ln r_1 = u_1$$
$$C_0 + D_0 \ln r_2 = u_2.$$

$$\Rightarrow \quad C_0 = \frac{\ln\left(\frac{r_2^{u_1}}{r_1^{u_2}}\right)}{\ln\left(\frac{r_1}{r_2}\right)}$$

$$D_0 = \frac{u_2 - u_1}{\ln\left(\frac{r_1}{r_2}\right)}$$

$$u(r,\theta) = \frac{\ln\left[\frac{r_2^{u_1}}{r_1^{u_2}}\right]}{\ln\left(\frac{r_1}{r_2}\right)} + \frac{u_2 - u_1}{\ln\left(\frac{r_1}{r_2}\right)} \ln r.$$

7. Because the general solutions are given by. (9)

$$U(r,\theta) = C_0 + D_0 \ln r + \sum_{m=1}^{\infty} \left(C_m r^m + \frac{D_m}{r^m}\right)(\cos m\theta + \sin m\theta)$$

So. when choosing $r = \frac{1}{r}$, we have

$$U\left(\frac{1}{r},\theta\right) = C_0 + D_0 \ln\frac{1}{r} + \sum_{m=1}^{\infty} \left(\frac{C_m}{r^m} + D_m r^m\right)(\cos m\theta + \sin m\theta)$$

$$= \widetilde{C_0} + \widetilde{D_0} \ln r + \sum_{m=1}^{\infty} \left(\widetilde{C_m} r^m + \frac{\widetilde{D_m}}{r^m}\right)(\cos m\theta + \sin m\theta)$$

where $\begin{cases} \widetilde{C_0} = C_0 \\ \widetilde{D_0} = -D_0 \\ \widetilde{C_m} = D_m \\ \widetilde{D_m} = C_m \end{cases}$.

So this is also <u>harmonic</u>!