PS3 Q9(3):

The principal part is

$$a u_{xx} + 2b u_{xy} + c u_{yy}$$

$$= u_{xx} + 2u_{xy} + 17 u_{yy}$$

with $a=1$, $b=1$, $c=17$.

Use the change of variables (week 3 Notes 3.3)

$$\begin{cases} x' = x \\ y' = -\frac{b}{a}x + y = -x + y \end{cases}$$

Using this change of variables

$$u_{xx} + 2u_{xy} + 17 u_{yy}$$

$$= a\left(u_{x'x'} + \left(\frac{ac-b^2}{a^2}\right) u_{y'y'}\right)$$

$$= u_{x'x'} + \frac{1\cdot 17 - 1^2}{1} u_{y'y'}$$

$$= u_{x'x'} + 16 u_{y'y'}$$

PS4 Q2:

We compute

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) V(x,t)$$

$$= \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) \left[\frac{\partial}{\partial x} \cdot u(x,t)\right]$$

$$= \frac{\partial}{\partial x}\left[\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u(x,t)\right]$$

$$= \frac{\partial}{\partial x}\left[u_{tt} - c^2 u_{xx}\right]$$

$$= \frac{\partial}{\partial x} \cdot 0 \quad \leftarrow \quad u \text{ satisfies wave equation because}$$

$$= 0$$

so v also satisfies wave equation.

PS4 Q5:

The equation can be factored as

$$\left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\underbrace{\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) u}_{} = 0$$

Denote by $\frac{\partial}{\partial x} + \frac{\partial}{\partial t} u = w$

we get 2 1st order PDEs

$$\begin{cases} \frac{\partial}{\partial x} - 4\frac{\partial}{\partial t} w = 0 & (1) \\ \frac{\partial}{\partial x} + \frac{\partial}{\partial t} u = w & (2) \end{cases}$$

Equation (1) has characteristics $\frac{dt}{dx} = -4$

Namely $\quad t = -4x + c$

So ① becomes an ODE along such characteristics

$$\frac{d}{dx} w(x, t(x)) = w_x + \frac{dt}{dx} w_t$$
$$= w_x - 4 w_t$$
$$= 0$$

Thus $\quad w(x,t) = f(c)$

using that $\quad c = 4x + t$, we get

$$w(x,t) = f(4x+t)$$

Next, we solve ②, namely

$$u_x + u_t = f(4x+t) \quad ③$$

This equation has characteristics $\frac{dt}{dx} = 1$

Namely $\quad t = x + \tilde{c}$

Along such characteristics

(and using $t = x + \tilde{c}$), we see

③ becomes an ODE

$$\frac{d}{dx} u(x, t(x)) = f(4x + x + \tilde{c})$$

$$\frac{d}{dx} u = f(5x + \tilde{c})$$

Integrate it with respect to x, we get

$$u = F(5x + \tilde{c}) + G(\tilde{c})$$

substituting back $\tilde{c} = t - x$, we get the general solutions:

$$u(x,t) = F(5x + t - x) + \tilde{G}(t-x)$$

$$= F(4x+t) + G(x-t) \quad (4)$$

for any F, G.

Next we determine F, G by initial conditions

First $u_t(x,t) = F'(4x+t) - G'(x-t)$ (5)

plug in $t=0$ to (4) and (5), we get

$$x^2 = u(x,0) = F(4x) + G(x) \quad (6)$$

$$e^x = u_t(x,t) = F'(4x) - G'(x) \quad (7)$$

Integrate (7) we get

$$e^x = \frac{1}{4}F(4x) - G(x) + D \quad (8).$$

Using (6) and (8), we can solve the algebraic equations and get F, G.

$$\frac{x^2+e^x-D}{\frac{5}{4}} = F(4x)$$

substituting $\frac{x}{4}$ for x, we get

$$F(x) = \frac{4}{5}\left[\frac{x^2}{16}+e^{\frac{x}{4}} - D\right]$$

and $G(x) = F(4x) - x^2$

$$= \frac{4}{5}(x^2+e^x - D) - x^2$$

$$= -\frac{1}{5}x^2 + \frac{4}{5}e^x - \frac{4}{5}D$$

plug in $x=0$ in ⑥ we get

$$F(0)+G(0) = 0 ,$$

Namely

$$\frac{4}{5}[0+e^0-D] - 0 + \frac{4}{5}e^0 - \frac{4}{5}D = 0$$

so $D = 1$

Thus $F(x) = \frac{4}{5}\left(\frac{x^2}{16}+e^{\frac{x}{4}}-1\right)$

$$G(x) = -\frac{1}{5}x^2 + \frac{4}{5}e^x - \frac{4}{5}$$

plug into the general solutions, get

$$\boxed{u(x,t) = \frac{4}{5}\left[\frac{(4x+t)^2}{16}+e^{\frac{4x+t}{4}}-1\right] - \frac{1}{5}(x-t)^2 + \frac{4}{5}e^{x-t} - \frac{4}{5}}$$

PS4 Q9:

First the general solution for wave equations we deduced in the lecture notes are

$u(x,t) = F(x+ct) + G(x-ct)$

for any F, G

Next we use the boundary conditions to determine F, G.

when $x-ct=0$, Namely $t = \frac{x}{c}$, we have

① $x^2 = u|_{x=ct} = F(x+x) + G(x-x) = F(2x)+G(0)$

when $x+ct=0$, Namely $t = -\frac{x}{c}$, we have

② $x^4 = u|_{x+ct} = F(x-x) + G(x+x) = F(0) + G(2x)$

plugin $x=0$ in ①, we get

③ $F(0) + G(0) = 0$

Replace x by $\frac{x}{2}$ in ①, we get

$F(x) = \frac{x^2}{4} - G(0)$

replace x by $\frac{x}{2}$ in ②, we get

$$G(x) = \frac{x^4}{16} - F(0)$$

so the solution is

$$u(x,t) = F(x+ct) + G(x-ct)$$

$$= \frac{(x+ct)^2}{4} - G(0) + \frac{(x-ct)^4}{16} - F(0)$$

using ③, $F(0) + G(0) = 0$,

so $$u(x,t) = \frac{(x+ct)^2}{4} + \frac{(x-ct)^4}{16} - (F(0)+G(0))$$

$$= \frac{(x+ct)^2}{4} + \frac{(x-ct)^4}{16}$$

PS 5 Q 2:

First consider solutions with separated variables

$$u(x,t) = X(x)T(t).$$

plug into the equation gives

$$X \cdot \ddot{T} - c^2 X'' T = 0$$

$$\frac{\ddot{T}}{cT^2} = \frac{X''}{X} = -\lambda \qquad (A)$$

The 2nd idenity in (*) together with the boundary conditions gives the eigenvalue problem

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0, \quad X'(\pi) = 0 \end{cases}$$

The general solutions for X are

$$X(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x).$$

It's derivative is

$$X'(x) = -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda} x) + C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$$

The first boundary condition gives

$$0 = X'(0) = C_2 \sqrt{\lambda},$$

This implies $C_2 = 0$ and $C_1 \neq 0$.

The second boundary condition then gives

$$0 = X'(\pi) = -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda} \pi)$$

Thus $\sqrt{\lambda} \pi = n\pi, \quad n = 0, 1, 2, \cdots$

So eigenvalues are $\lambda_n = n^2, \quad n = 0, 1, 2 \cdots$

eigenfunctions are $X_n(x) = \cos(nx)$

For $n \geq 1$, knowing λ_n, we can solve

$$\frac{\ddot{T}}{c^2 T} = T \quad \text{and get}$$

$$T_n(t) = a_n \cos(cnt) + b_n \sin(cnt)$$

For $n = 0$, i.e. $\lambda = 0$, we also have we have $X_0(x) = \cos 0 = 1$

and solving $T_0'' = 0$ gives $T_0 = a_0 + b_0 t$.

So the general solutions is given by

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x) T_n(t)$$

$$= a_0 + b_0 t + \sum_{n=1}^{\infty} a_n \cos(nx) \cos(cnt) + \sum_{n=1}^{\infty} b_n \cos(nx) \sin(cnt)$$

Its time derivative is

$$u_t(x,t) = b_0 - \sum_{n=1}^{\infty} c n \cdot a_n \cos(nx) \sin(cnt) + \sum_{n=1}^{\infty} c n \cdot b_n \cos(nx) \cos(cnt)$$

plug in $t = 0$, we get

$$0 = u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$\frac{1}{2} + \frac{\cos 2x}{2} = u_t(x,0) = b_0 + \sum_{n=1}^{\infty} c \cdot n \cdot b_n \cos(nx)$$

Thus we can determine the coefficients.

using that $\sin mx$ $\cos nx$ are

independent if $m \neq n$, we get

$a_n = 0$ for all n

$b_n = 0$ for all n except for $n = 0$ or 2

and $b_0 = \frac{1}{2}$, $c \cdot 2 \cdot b_2 = \frac{1}{2} \Rightarrow b_2 = \frac{1}{4c}$

So the solution to this Question is

$$u(x,t) = \frac{1}{2} t + \frac{1}{4c} \cos(2x) \sin(2ct)$$

PS 5 Q4:

The Fourier series is (as defined in Week 6 notes)

$$\hat{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{\pi}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{\pi}\right)$$

with $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\sin x|\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{1}{\pi}\cdot(-\cos x)\Big|_0^{\pi}$$

$$= \frac{2}{\pi}$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi x}{\pi}\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

We compute $\int_0^{\pi} \sin x \cos nx\,dx = -\cos x \cos nx\Big|_0^{\pi} - \int_0^{\pi}(-\cos x)(-n\sin nx)\,dx$

$$= (-1)^n + 1 - n\int_0^{\pi} \cos x \sin nx\,dx$$

$$= (-1)^n + 1 - n\sin x \sin nx\Big|_0^{\pi} + n^2\int_0^{\pi} \sin x \cos nx\,dx$$

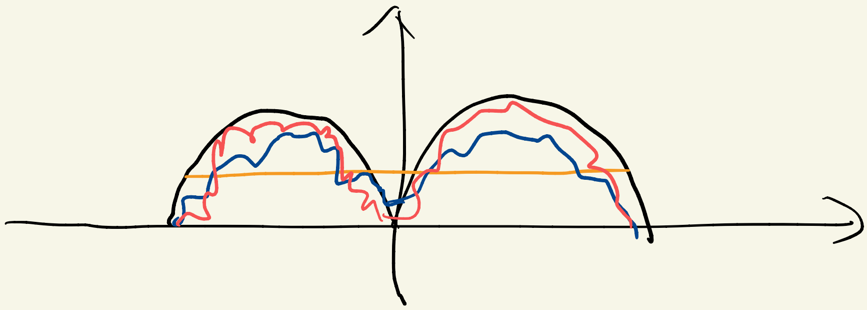
$$= 1 + (-1)^n + n^2\int_0^{\pi} \sin x \cos nx\,dx$$

Thus $\int_0^{\pi} \sin x \cos nx\,dx = \frac{-1-(-1)^n}{n^2-1}$

and $a_n = \frac{2[-1-(-1)^n]}{\pi\cdot(n^2-1)}$

Noticing $f(x) = |x|$ is even, so all $b_n = 0$, $n = 1, 2, \cdots$

Thus $\hat{f}(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \cdot \frac{2[-1-(-1)^n]}{\pi \cdot (n^2-1)} \cos(nx)$



PS 6 Q1:

using the formula (3.6) in week 6 notes for inhomogeneous problem with

$\psi(x) = \sin x$
$f(x) = 0$ and $c = 1$
$g(x) = \sin x$

we get

$$u(x,t) = \frac{0+0}{2} + \frac{1}{2}\int_{x-t}^{x+t} \sin s \, ds + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} \sin r \, dr \, ds$$

$$= \frac{\cos(x-ct) - \cos(x+ct)}{2} + \frac{1}{2}\int_0^t [\cos(x-t+s) - \cos(x+t-s)] \, ds$$

$$= \frac{\cos(x-ct) - \cos(x+ct)}{2} + \frac{\sin x - \sin(x-t)}{2}$$

$$-\frac{1}{2}\cdot\left[-\sin x + \sin(x+t)\right]$$

$$=\frac{\cos(x-ct)-\cos(x+ct)}{2}+\sin x-\frac{\sin(x-ct)+\sin(x+ct)}{2}$$

PS 6 Q4:

For this Question, we can directly apply the formula in section 3 of Week 5 notes

with $L=\pi$

$$f = b\sin x + 2b\sin(2x)$$

$$g = \sin x$$

thus $a_n=\frac{2}{\pi}\int_0^L [b\sin x + 2b\sin(2x)]\sin(nx)dx$

$$b_n=\frac{2}{n\pi c}\int_0^L \sin x\sin(nx)\,dx$$

Using the orthogonality of $\sin nx$ and $\sin mx$

we see all $a_n=0$ except for $n=1,2$,

and $a_1=b$, $a_2=2b$

all $b_n=0$ except for $n=1$

and $b_1=\frac{1}{c}$

so

$$u(x,t) = b \cdot \sin x \cos(ct) + 2b \sin(2x)\cos(2ct)$$
$$+ \tfrac{1}{c} \sin x \sin(ct)$$

PS 6 Q 8:

- we differentiate $E[u](t)$ with respect to t and get

$$\frac{d}{dt} E[u](t)$$

$$= \frac{d}{dt} \frac{1}{2} \int_{-\infty}^{\infty} \left[u_t^2 + \partial^2 u_x^2 \right] dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d}{dt} [u_t^2 + \partial^2 u_x^2] dx$$ ✓ chain rule

$$= \frac{1}{2} \int_{-\infty}^{\infty} [2 \cdot u_t \cdot u_{tt} + 2\partial^2 \cdot u_x \cdot u_{xt}] dx$$

$$= \int_{-\infty}^{\infty} u_t \cdot u_{tt} dx + \partial^2 \int_{-\infty}^{\infty} u_x \cdot u_{xt} dx$$ ✓ integration by parts

$$= \int_{-\infty}^{\infty} u_t \cdot u_{tt} dx + \partial^2 u_x \cdot u_t \Big|_{x=-\infty}^{\infty} - \partial^2 \int_{-\infty}^{\infty} u_{xx} \cdot u_t dx$$

$$= \int_{-\infty}^{\infty} u_t \cdot u_{tt} dx - \int u_t \cdot \partial^2 u_{xx} dx$$ ✓ using compact support.

$$= \int_{-\infty}^{\infty} u_t [u_{tt} - \partial^2 u_{xx}] dx$$

$= \int_{-\infty}^{\infty} u_t \cdot c u_t$ ✓ using the PDE $u_{tt} - \partial^2 u_{xx} = c \cdot u_t$

$= c \cdot \int_{-\infty}^{\infty} u_t^2$

≤ 0 (because $c < 0$ and $\int_{-\infty}^{\infty} u_t^2 \geq 0$)

so the energy is non-increasing.

- For the second part of the question,

suppose u_1 and u_2 are 2 solutions to

$$\begin{cases} u_{tt} - \partial^2 u_{xx} - c u_t = F(x), \quad c < 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases} \quad (\ast\ast)$$

then $v = u_1 - u_2$ is a solution to

$$\begin{cases} v_{tt} - \partial^2 v_{xx} - c v_t = 0 \\ v(x,0) = 0, \quad v_t(x,0) = 0 \end{cases}$$

Notice the initial condition gives

$E[v](0) = \frac{1}{2} \int 0 + \partial^2 \cdot 0 = 0$

and the first part shows us $E[v](t)$ is non-increasing.

But because $E[v](t) \geq 0$ for all t,

so we must have $E[v](t) \equiv 0$

Namely $V_x \equiv 0$ and $V_t \equiv 0$.

$$V(x,t) = \int_0^t V_t(x,s)\,ds + V(x,0)$$
$$= 0 + 0 = 0$$

So $V \equiv 0$ and thus $U_1 \equiv U_2$

and the solution to $(\ast\ast)$ is unique.

PS 7 Q 4:

The general solutions to Laplace equation in a disc of radius r_* is (as deduced in Week 8 Notes)

$$u(x,t) = a_0 + \sum_{m=1}^{\infty} r^m (a_m \cos m\theta + b_m \sin m\theta)$$

with boundary condition

$$f(\theta) = u(r_*, \theta) = a_0 + \sum_{m=1}^{\infty} r_*^m (a_m \cos m\theta + b_m \sin m\theta)$$

Here $f(\theta) = \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

so using the fact that $\cos(m\theta)$ and $\sin n\theta$ are independent, we observe that

$$a_0 = \frac{1}{2}, \quad r_*^2 \cdot a_2 = -\frac{1}{2} \Rightarrow a_2 = -\frac{1}{2r_*^2}$$

all other a_n and b_n are zero.

so
$$u(x,t) = \frac{1}{2} - \frac{r^2}{2r_*^2}\cos(2\theta)$$