In this manner we can recast questions about the orbits of T (and also the logistic map f_4) into questions about sequences of L's and R's.

$f_4(x) = 4x(1-x)$

Our discussion embodies the idea behind:

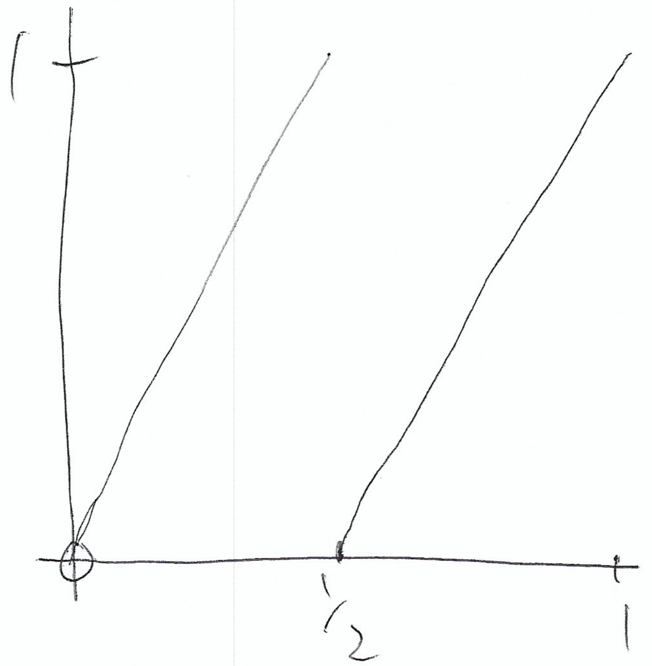
<u>Symbolic dynamics</u>: Given a dynamical system, we <u>encode</u> the orbit of a given initial condition x_0 by a sequence of symbols (i.e. an infinite sequence where each term in the sequence is drawn from a given finite set (often the finite set is called an 'alphabet')). This can be used to study the original dynamical system.

Important case The doubling map

$$D(x) = 2x \pmod 1$$

i.e. $D : [0,1) \longrightarrow [0,1)$

$$D(x) = \begin{cases} 2x & \text{if } x \in [0, \tfrac{1}{2}) \\ 2x - 1 & \text{if } x \in [\tfrac{1}{2}, 1) \end{cases}$$



Remarks

- We say that $x = y \pmod 1$ if $x - y$ is an integer

- D has a single fixed point, namely the one at $x = 0$ (note $1 \notin [0,1)$, so 1 is not a fixed point)

- There is a single 2-cycle, $\{1/3, 2/3\}$

- There are two 3-cycles, namely $\{1/7, 2/7, 4/7\}$, and $\{3/7, 6/7, 5/7\}$

- In general, a point x has period n if $D^n(x) = x$

ie. $2^n x = x \pmod 1$

ie. $2^n x - x = p$, where $p \in \mathbb{Z}$

ie. $x = \dfrac{p}{2^n - 1}$

Thus the set of period-n points is

$$\mathrm{Per}(n) = \left\{ \frac{p}{2^n - 1} : p = 0, 1, \ldots, 2^n - 2 \right\}$$

In particular, there are precisely $2^n - 1$ points of period n.

If we use symbols 0 and 1 in our symbolic coding (instead of 'L' and 'R') then for example the period-4 point $\frac{1}{5}$ corresponds to the (periodic) symbol sequence 0011001100110011...

Note $\frac{1}{5}$ lies in the 4-cycle $\{\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}\}$

Let $S := \{(b_1, b_2, b_3, \ldots) : b_i \in \{0,1\}$ for all $i \in \mathbb{N}\}$

Define $\sigma : S \to S$ by

$$\sigma(b_1 b_2 b_3 \ldots) = b_2 b_3 b_4 \ldots$$

i.e. $\sigma\left((b_i)_{i=1}^{\infty}\right) = (b_{i+1})_{i=1}^{\infty}$

This is called the <u>shift map</u>.

Define $b : [0,1) \to S$ by

$$b(x) = b_1 b_2 b_3 \ldots$$

where $b_i = \begin{cases} 0 & \text{if } D^{i-1}(x) \in [0, \tfrac{1}{2}) \\ 1 & \text{if } D^{i-1}(x) \in [\tfrac{1}{2}, 1) \end{cases}$

It can be shown that if

$b(x) = b_1 b_2 b_3 \ldots$ then

$$x = \sum_{i=1}^{\infty} b_i/2^i = \frac{b_1}{2} + \frac{b_2}{4} + \frac{b_3}{8} + \ldots \qquad (*)$$

i.e. the "coding sequence" $b_1 b_2 b_3 \ldots$ for x gives a <u>binary expansion</u> (i.e. <u>base-2 expansion</u>) for x

<u>Example</u> We have seen that

$b(1/5) = 001100110011\ldots$

Note that $(*)$ in this case gives

RHS of $(*)$ =

$$\frac{0}{2} + 0/4 + 1/8 + 1/16 + 0/32 + 0/64 + 1/128 + 1/256 + \ldots$$

$$= \left(\frac{1}{8} + \frac{1}{16}\right)\left(1 + \frac{1}{2^4} + \left(\frac{1}{2^4}\right)^2 + \left(\frac{1}{2^4}\right)^3 + \dots\right)$$

$$= \frac{3}{16} \sum_{i=0}^{\infty} \left(\frac{1}{16}\right)^i$$

$$= \frac{3}{16} \frac{1}{1 - 1/16}$$

$$= \frac{3}{16} \frac{16}{15}$$

$$= \frac{3}{15}$$

$$= \frac{1}{5} \checkmark$$

<u>Notation</u> For periodic sequences $b_1 b_2 \dots b_n\, b_1 b_2 \dots b_n\, b_1 b_2 \dots b_n \dots$ we shall often use the notation

$\overline{b_1 b_2 \dots b_n}$ to denote this sequence.

Note that we can introduce a map $h : S \to [0,1]$ defined by

$$h(b_1 b_2 b_3 \ldots) = \sum_{i=1}^{\infty} \frac{b_i}{2^i}$$

<u>Claim</u> We then have that

$$D \circ h = h \circ \sigma \qquad (**)$$

In other words,

$$\begin{tikzcd} S \arrow[r, "\sigma"] \arrow[d, "h"'] & S \arrow[d, "h"] \\ {[0,1)} \arrow[r, "D"] & {[0,1)} \end{tikzcd}$$

is a commutative diagram.

To check the Claim, note that the RHS of $(**)$ is:

$$h\circ\sigma(b_1 b_2 b_3 \ldots)$$

$$= h(b_2 b_3 b_4 \ldots)$$

$$= \frac{b_2}{2} + \frac{b_3}{2^2} + \frac{b_4}{2^3} + \ldots,$$

whereas the LHS of $(**)$ is:

$$D\circ h(b_1 b_2 b_3 \ldots)$$

$$= D\left(\frac{b_1}{2} + b_2/4 + b_3/8 + \ldots\right)$$

$$= b_1 + \frac{b_2}{2} + b_3/4 + \frac{b_4}{8} + \ldots \pmod 1$$

$$= \frac{b_2}{2} + b_3/4 + b_4/8 + \ldots$$

$$= \text{RHS of } (**) \quad \checkmark$$

Note that h is not a bijection, since it is not injective (i.e. it is not one-to-one), as we see with the following example:

$$h(0\bar{1}) = h(0111111\ldots)$$

$$= \frac{0}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

$$= \sum_{i=2}^{\infty} \left(\frac{1}{2}\right)^i$$

$$= \frac{1/4}{1 - 1/2}$$

$$= \frac{1/4}{1/2}$$

$$= \frac{1}{2},$$

but also

$$h(1\overline{0}) = h(100000\ldots)$$

$$= \frac{1}{2} + \frac{0}{2^2} + \frac{0}{2^3} + \frac{0}{2^4} + \ldots$$

$$= \frac{1}{2}$$

So $h(0\overline{1}) = h(1\overline{0})$, so h is <u>not</u> one-to-one.

(Compare this to the familiar fact that in decimal notation, the number 1 has the two representations

$$1 = 1.00000\ldots$$

$$= 0.99999\ldots$$

Chaos (what might it mean?)

The doubling map has the property that its periodic points are 'dense' in $[0,1)$, i.e. for any $x \in [0,1)$ and any $\varepsilon > 0$ there exists a periodic point y with $|x-y| < \varepsilon$

(Recall that rational numbers of the form $\frac{p}{2^n-1}$ are periodic (of period n) under D)

The doubling map D also has the property that there is a dense orbit, i.e. there is a point $x_0 \in [0,1)$ such that

$\mathcal{O}(x_0) = \{ D^i(x_0) : i = 0,1,2,3,\dots \}$

is 'dense' in $[0,1)$

i.e. For any $x \in [0,1)$ and any $\varepsilon > 0$ there exists $i \geq 0$ such that

$$| D^i(x_0) - x | < \varepsilon$$