MTH6107 Chaos & Fractals

Solutions 4

EXAM QUESTION: the questions below are based on the various parts of Question 2 on the January 2023 exam paper

Suppose $a \geq 2$, and that the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - a$.

Exercise 1. Determine all fixed points of f, and determine whether each fixed point is attracting or repelling, taking care to justify your answer.

Fixed points $x = f(x)$ must satisfy $x^2 - x - a = 0$, and the two roots $x^+ = \frac{1}{2}(1 + \sqrt{1+4a})$ and $x^- = \frac{1}{2}(1 - \sqrt{1+4a})$ are both real and distinct, so are fixed points of f.

Now $f'(x) = 2x$, so $f'(x^+) = 1 + \sqrt{1+4a} > 1$, and $f'(x^-) = 1 - \sqrt{1+4a} < -1$ (since $a \geq 2$), so both fixed points are repelling (as their multipliers are larger than 1 in modulus).

Exercise 2. Determine all 2-cycles for f, and determine whether each 2-cycle is attracting or repelling, taking care to justify your answer.

Period-2 points x satisfy $f^2(x) = x$, i.e. $(x^2 - a)^2 - a = x$, i.e. are solutions of the quartic equation $x^4 - 2ax^2 - x + a^2 - a = 0$.

Using the fact that fixed points are period-2 points we know that $f(x) - x = x^2 - x - a$ is a factor of the above quartic polynomial, which we can then factorise to see that period-2 points are solutions of the equation

$$(x^2 - x - a)(x^2 + x + 1 - a) = 0\,.$$

The quadratic equation $x^2 + x + 1 - a = 0$ has solutions $y^+ = \frac{1}{2}(-1 + \sqrt{4a-3})$ and $y^- = \frac{1}{2}(-1 - \sqrt{4a-3})$ (these are both real, since $a \geq 1$), so f has precisely one 2-cycle, namely $\{y^+, y^-\}$.

The multiplier of the 2-cycle is

$$f'(y^+)f'(y^-) = (-1 + \sqrt{4a-3})(-1 - \sqrt{4a-3}) = 1 - (4a-3) = 4 - 4a\,,$$

and $4 - 4a < -1$, since $a \geq 2$, therefore this 2-cycle is repelling.

Exercise 3. Give one example of an eventually fixed point that is not itself a fixed point, and one example of an eventually periodic point of least period 2 that is not itself a periodic point.

In general $f(x) = f(-x)$, and in particular if x is a fixed point then $f(-x) = f(x) = x$, so that $-x$ is an eventually periodic point. So $-x^+ = -\frac{1}{2}(1+\sqrt{1+4a})$ is an eventually periodic point that is not itself a fixed point (as is $-x^- = -\frac{1}{2}(1-\sqrt{1+4a})$).

By the same reasoning as above, we see that $-y^+ = -\frac{1}{2}(-1+\sqrt{4a-3})$ is an eventually periodic point of least period 2 that is not itself a periodic point (as is $-y^- = -\frac{1}{2}(-1-\sqrt{4a-3})$).

Exercise 4. If $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x^2 + a$, determine whether there is a topological conjugacy from f to g, taking care to justify your answer.

There is not a topological conjugacy, since f has two fixed points (by Exercise 1 above) whereas g does not have any fixed points (since the fixed point equation $g(x) - x = 0$ has no solutions, since the quadratic equation $x^2 - x + a = 0$ has only non-real solutions $\frac{1}{2}(1 \pm \sqrt{1-4a})$, since $1 - 4a < 0$).

Exercise 5. If $F : \mathbb{R} \to \mathbb{R}$ and and $G : \mathbb{R} \to \mathbb{R}$ are defined by $F(x) = x - a$ and $G(x) = x + a$, determine whether there is a topological conjugacy from F to G, taking care to justify your answer.

There is a topological conjugacy: taking the homeomorphism $H(x) = -x$ we see that $H(F(x)) = -F(x) = -(x-a) = -x + a = H(x) + a = G(H(x))$.