MTH5104 CONVERGENCE AND CONTINUITY (2023-2024)
QUIZ 2 (WEEK 6)

Upload your solutions as a single PDF as an answer to the QMPlus quiz. Make sure that your scans are legible before uploading them.

(1) Prove *directly from the definition* that:

$$\frac{3}{n+1} \to 0.$$

(5 marks)

(2) Prove that:

$$\frac{n^2+3n}{2n^2+1} \to \frac{1}{2}.$$

You may use any results from lectures, but you *must* state them clearly.

(5 marks)

Solutions.

(1) Given $\epsilon > 0$ choose $N \in \mathbb{N}$ such that $N \geq 3/\epsilon$ (this exists by the Archimedean property). This implies that $3/N \leq \epsilon$. Then for all $n > N$ we have

$$\frac{3}{n+1} < \frac{3}{n} < \frac{3}{N} \leq \epsilon$$

and so

$$\left|\frac{3}{n+1}\right| = \frac{3}{n+1} < \epsilon$$

as required.

(2) We rewrite this expression as:

$$\frac{n^2+3n}{2n^2+1} = \frac{1+\frac{3}{n}}{2+\frac{1}{n^2}}.$$

We have seen in lectures that $\frac{1}{n} \to 0$. A result from lectures says that if $x_n \to 0$ then $cx_n \to 0$ for all $c \in \mathbb{R}$. We conclude that $\frac{3}{n} \to 0$.

On the other hand we have

$$\left|\frac{1}{n^2}\right| \leq \left|\frac{1}{n}\right|$$

for all $n \in \mathbb{N}$. A result from lectures (dominated convergence) says that if $y_n \to 0$ and $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$, then $x_n \to 0$. Since $\frac{1}{n} \to 0$ we conclude that $\frac{1}{n^2} \to 0$.

A result from lectures says that if $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$. We conclude that

$$1+\frac{3}{n} \to 1, \qquad 2+\frac{1}{n^2} \to 2.$$

Finally a result from lectures says that if $x_n \to x$ and $y_n \to y$ with $y \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$, then $x_n/y_n \to x/y$. Applying this to the above, we conclude that

$$\frac{n^2+3n}{2n^2+1} \to \frac{1}{2}.$$