MTH744

Midterm test and solutions

November 2023

MTH744U/P - DYNAMICAL SYSTEMS

MID-TERM TEST 2023-24 (1 hour; SEA 1.5 hours)

Programmable calculators not allowed.

You can submit your rough working as evidence of your methods, but it is not obligatory.

10.15am, Thursday Nov 9th, 2023.

Setter: D.K. Arrowsmith

Question 1 Identifying phase portraits

Which of the following ODEs on $\mathbb{R}$ give a phase portrait which is qualitatively the same as the one illustrated below?

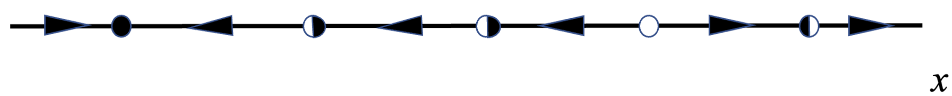


Figure 1: Phase portrait on $\mathbb{R}$.

(a) $\dot{x} = (4-x)^2(x+3)^2(x-2)(x+4)$ [ ]

(b) $\dot{x} = -x(x-2)^2(x+3)^2(x-4)$ [ ]

(c) $\dot{x} = (x+2)^2(x+3)(x^2-9)(x-4)$ [ ]

(d) None of above. [ ]

Question 2 Classification of a fixed point

The fixed point at $x = 0$ of the ODE $\dot{x} = x(x - \sinh(x))^3$, $x \in \mathbb{R}$, satisfies which of the following options

(a) a saddle node with instability for $x > 0$ [ ]

(b) a saddle node with instability for $x < 0$ [ ]

(c) unstable fixed point [ ]

(e) stable fixed point [ ]

(f) None of the above. [ ]

Question 3 Phase portraits on the circle

For which of the functions $g : \mathbb{S} \to \mathbb{R}$ (see the choices below) does the system

$$\dot{\theta} = g(\theta),$$

$\theta \in \mathbb{S}$ have exactly four fixed points with one unstable, one asymptotically stable, and two saddle-nodes.

(a) $g(\theta) = \sin(2\theta)$ [ ]

(b) $g(\theta) = \sin(\theta)(1 - \sin(\theta))$ [ ]

(c) $g(\theta) = \sin(\theta)(1 - \cos^2(\theta))$ [ ]

(d) None of the above.

Question 4 Fixed Point Sets and Bifurcations

Consider the system

$$\dot{x} = x^4 - rx^3 - x^2 + r^2,$$

where $x \in \mathbb{R}$ and r is a real parameter. The system has one line of fixed points in the xr-plane given by $x = r$. Find the complete set of fixed point curves in in the xr-plane and use that information to decide on the following statements as true or false:

(a) there are at most 3 bifurcation points [ ]

(b) there are no transcritical bifurcations [ ]

(c) there are no pitchfork bifurcations [ ]

(d) there are no saddle-node bifurcations [ ]

(e) None of the above. [ ]

Question 5 Finding Bifurcation Types

For the system

$$\dot{x} = rx + x\frac{(1 + r + x)}{(1 + x^2)},$$

where $x \in$R and r is a real parameter. Is there a bifurcation point of type **(?)** at the point **(?)** in the following choices?

(a) saddle node at $(x, r) = (0, 0)$; [ ]

(b) pitchfork at $(x, r) = (0, \frac{1}{2})$; [ ]

(c) transcritical at $(x, r) = (0, -\frac{1}{2})$; [ ]

(d) None of these [ ]

Question 6 Fixed points of planar linear systems

Describe the type of the fixed point at the origin $(x, y) = (0, 0) \in \mathbb{R}^2$ for the linear system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix},$$

when the matrix

(a) A= $\begin{bmatrix} 1 & 6 \\ 3 & 7 \end{bmatrix}$ [ ]

(b) A= $\begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$ [ ]

(c) A= $\begin{bmatrix} -1 & -1 \\ -1 & -3 \end{bmatrix}$ [ ]

(d) A= $\begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$. [ ]

Your answers should be chosen from:

saddle (SA); unstable node (UN); stable node (SN); unstable spiral (US); stable spiral (SS); unstable improper node(UIN); stable improper node (SIN); unstable (U); stable(ST); stable but not asymptotically stable (SNAS).

Q1

Phase portrait has 5 distinct points

(a) $\dot{x} = (4-x)^2 (x+3)^2 (x-2)(x+4)$

Fixed points at
$x = -4, -3, 2, 4$ (4 FPs)

(b) $x = 0, 2, -3, 4$ (4 FPs).

(c) $x = -2, -3, 3, 4$ (4 FPs).

(d) None of the above ✓

Other observations - needs 3 double roots. for the saddle nodes

Q2.

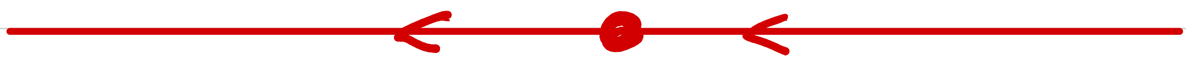
$$\dot{x} = x(x - \sinh(x))^3$$

Sinh x $= x + \frac{x^3}{3!} \ldots$

$$\sinh(x) = x + \frac{x^3}{3!} + \ldots$$

$$\dot{x} = x\left(-\frac{x^3}{3!} + \ldots\right)^3 = -\frac{x^{10}}{(6)^3} + \text{h.o.t.}$$

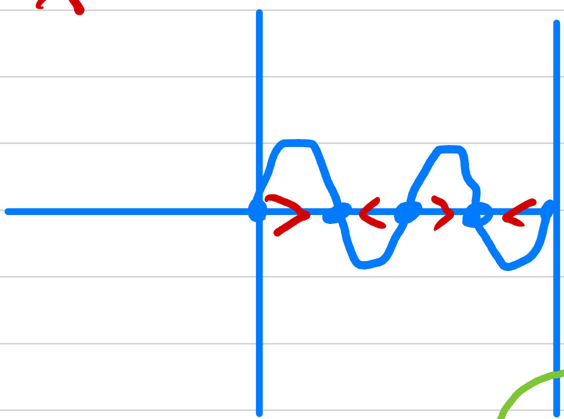
$\therefore$ $x = 0$ fixed pt has local approx $\dot{x} = -x^{10}/(6)^3$



$x = 0$

(b) saddle node with instability for $x < 0$.

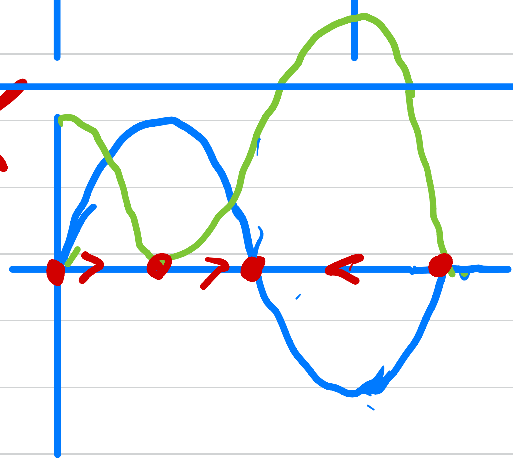
Q3 (a) $f(\theta) = \sin 2\theta$



4 fps
2 stable /
2 unstable
(not s-n s)

4 FPs, but no saddle nodes

(b) $f(\theta) = \sin\theta(1 - \sin\theta)$



equiv

3FPs! ✗

(c) $f(\theta) = \sin\theta\,(1-\cos^2\theta)$

$= \sin\theta\,(1-\cos\theta)(1+\cos(\theta))$

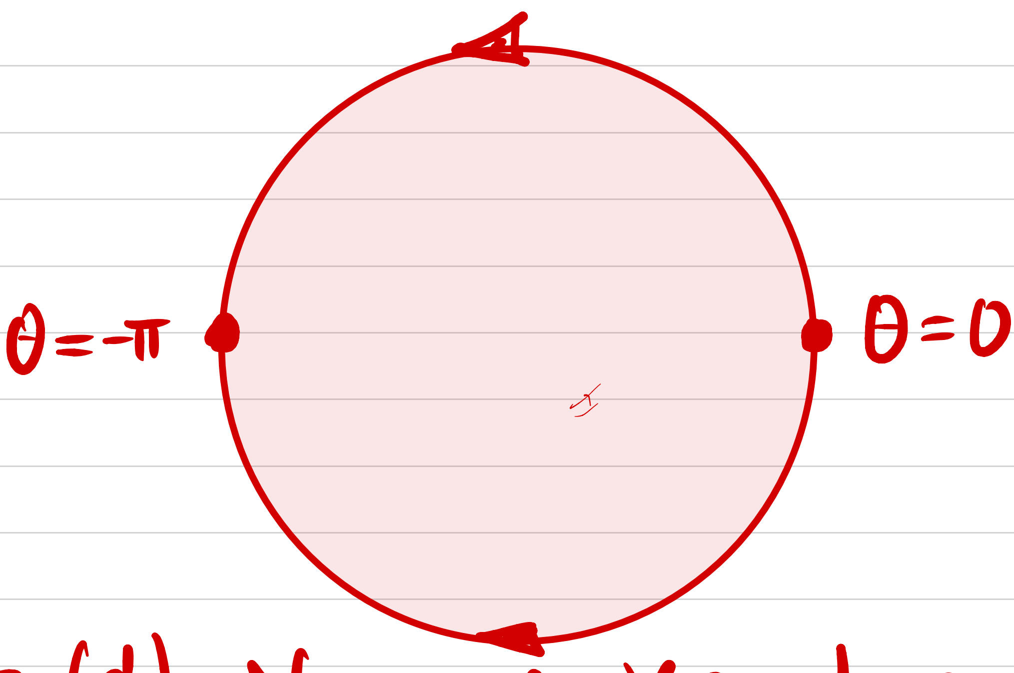
>0 except for $\theta = 0$ when $=0$

>0 except for $\theta = \pi$ when $=0$.

$\therefore \operatorname{Sgn}(\sin\theta) = \operatorname{sgn}(f(\theta))$ except for "added" zeroes at $\theta = 0$, $\theta = \pi$!

So no change from $\dot{\theta} = \sin\theta$!

i.e.



Choice (d) None of the above.

OR

$$\dot{\theta} = \sin\theta\,(1-\cos^2\theta)$$
$$= \sin\theta\,(\sin^2\theta)$$
$$= \sin^3(\theta)$$

$$\operatorname{sgn}(\sin^3\theta) = \operatorname{sgn}(\sin\theta)$$

So phase portraits of

$\dot{\theta} = \sin\theta$ and $\dot{\theta} = \sin^3(\theta)$

are identical.

As are on $\mathbb{R}$

$\dot{x} = x$ & $\dot{x} = x^3$

$\dot{x} = \sinh(x)$ $\quad \dot{x} = \sinh^3(x)$

$\dot{x} = f(x)$ & $\dot{x} = [f(x)]^3$

what abt

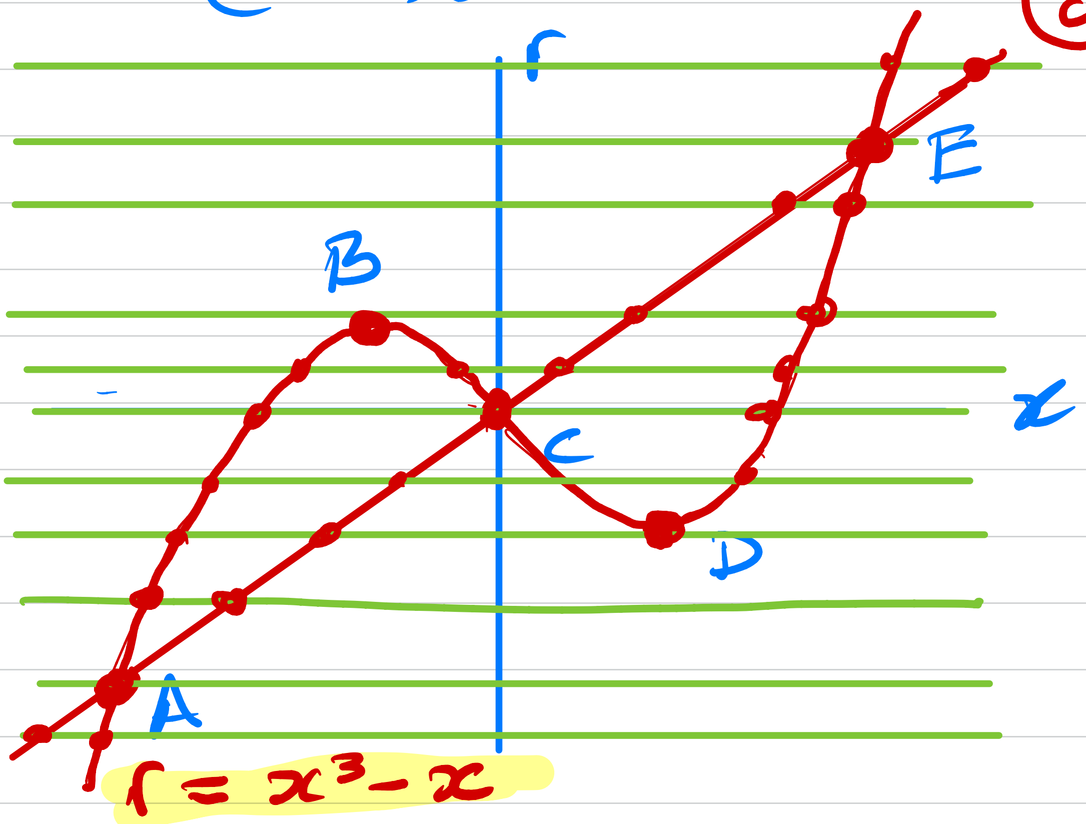
$\dot{x} = f(x)$ and $\dot{x} = [f(x)]^2$?

Q4. $\dot{x} = x^4 - r x^3 - x^2 + r^2$

$= (x-r)x^3 - (x-r)(x+r)$

$= (x-r)(x^3 - x - r)$

FPs: $\dot{x} = 0$



Fixed pt structure changes at A, B, C, D, E in the xr plane for r = constant. A C E potential Transcritical

BD potential saddle-node

(a) there are 5 ✗

(b) potentially 3 TB ✗

(c) no pitchfork ✓

(d) potentially 2 SN ✗

(e) none of the others ✗

Q5

$$\dot{x} = \overbrace{rx + x(1+r+x)/(1+x^2)}^{= f(x,r)}$$

Note $x=0$ is a fixed pt for all $r \in \mathbb{R}$

Expanding in powers of x:

$$= rx + x(1+r+x)(1-x^2+\dots)$$

$$= rx + x + rx + x^2 - x^3 - x^2 r - x^4 + \dots$$

$$= (2r+1)x + x^2 + \dots$$

At $x=0$, there is a bifurcation pt for $2r+1=0$, $\boxed{\text{Note } f_x'(x,r)\big|_{x=0} = 2r+1}$

Let $\mu = r + \frac{1}{2}$, then

$$\dot{x} = \boxed{2\mu x + x^2} - \frac{3}{2}x^3 + \mu x^3.$$

Transcritical bifurcation at $(x,r) = (0, -\frac{1}{2})$

$A \neq 0, C \neq 0$
$B = 0$
Trans Bif.

Note
$$\frac{1}{1+x^2} = 1 - x^2 + x^4$$

CHK "Binomial Exp Formula"

Q6 $\dot{\underline{x}} = \underline{A}\,\underline{x}$, $\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

(a) $A = \begin{bmatrix} 1 & 6 \\ 3 & 7 \end{bmatrix}$ $\quad$ $\text{Det} = 7 - 18 < 0$

λ_1, λ_2 opposite sign

saddle – unstable

(b) $A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix}$ $\quad$ $\text{Tr} = 7$ $\quad$ $\text{Det} = 16$

$\lambda_1, \lambda_2 = \left(7 \pm \sqrt{49 - 64}\right)/2$

$= \left(7 \pm i\sqrt{15}\right)/2$

unstable spiral – unstable

(c) $A = \begin{bmatrix} -1 & -1 \\ -1 & -3 \end{bmatrix}$ $\quad$ $\text{Tr} = -4$, $\text{Det} = 2$

$\lambda_1, \lambda_2 = \dfrac{-4 \pm \sqrt{8}}{2}$

stable node – stable and AS

(d) $A = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$ $\quad$ $\lambda_1 = 0$, $\lambda = -2$

stable fixed point – not AS

$\dot{y} < 0$

Note on (d)

(d) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\dot{x} = 0$, $\dot{y} = x - 2y$

$\Downarrow$

$x = \text{const}$

Not asymptotically stable

$\dot{y} > 0$