Queen Mary University of London

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MTH6107 Chaos & Fractals
MID-TERM TEST SOLUTIONS

Date: 8th November 2023 *Time:* 10am

Complete the following information:

Name	
Student Number (9 digit code)	

The test has SIX questions. You should attempt ALL questions. Write your calculations and answers in the space provided. Cross out any work you do not wish to be marked.

Question	Marks
1	
2	
3	
4	
5	
6	
Total Marks	

Nothing on this page will be marked!

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Question 1.
Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - 6$.

(a) Find all fixed points of f, and determine whether they are attracting or repelling.

(b) Determine the points of least period 2 for f.

[*25 marks*]

Answer 1.
(a) The fixed points are 3 and -2, the unique solutions of

$$0 = f(x) - x = x^2 - x - 6 = (x-3)(x+2)\,.$$

Both points are repelling, since $f'(x) = 2x$, so $|f'(3)| = 6 > 1$, and $|f'(-2)| = 4 > 1$.

(b) The points of least period 2 are $(-1+\sqrt{21})/2$ and $(-1-\sqrt{21})/2$.
These are the roots of the quadratic $q(x) = x^2 + x - 5$, and we note the factorisation

$$f^2(x) - x = (x^2-6)^2 - 6 = x^4 - 12x^2 - x + 30 = (x^2 - x - 6)(x^2 + x - 5) = (f(x) - x)q(x)\,.$$

Answer 1. (*Continue*)

Question 2.

Determine the basin of attraction for the fixed point 1 of the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 2x + 2$.

[*15 marks*]

Answer 2.

The basin of attraction is the open interval $(0, 2)$.

Note that $f(x) = (x-1)^2 + 1$, so if $x \in (0, 2)$ then the distance between 1 and $f^n(x)$ decreases monotonically to limit 0.
If $|x - 1| > 1$ then $f^n(x) \to \infty$ as $n \to \infty$.
The point 2 is a repelling fixed point, and the point 0 satisfies $f(0) = 2$, and therefore $f^n(0) = 2$ for all $n \geq 1$.

(Note that this example resembles the case $g(x) = x^2$ studied in lectures - in fact f and g are topologically conjugate, via the homeomorphism $h(x) = x - 1$).

Answer 2. (*Continue*)

Question 3.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1, and that $\{x_0, x_1, x_2, x_3, x_4, x_5\}$ is a 6-cycle with $f'(x_i) = (-2)^{i-3}$ for $0 \le i \le 5$. Determine, with justification, whether this cycle is attracting or repelling.

[*15 marks*]

Answer 3.

The cycle is attracting.

Justification: The multiplier $(f^6)'(x_0) = \prod_{i=0}^{5} f'(x_i)$, and we have that $|f'(x_0)| = 1/8$, $|f'(x_1)| = 1/4$, $|f'(x_2)| = 1/2$, $|f'(x_3)| = 1$, $|f'(x_4)| = 2$, $|f'(x_5)| = 4$, so

$$|(f^6)'(x_0)| = \prod_{i=0}^{5} |f'(x_i)| = \frac{1}{8} \times \frac{1}{4} \times \frac{1}{2} \times 1 \times 2 \times 4 = \frac{1}{8} < 1\,,$$

therefore by a result from the module we know this 6-cycle is attracting.

Answer 3. (*Continue*)

Question 4.

Suppose the diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 1 - x + \frac{1}{2}\cos x$. How many fixed points does f have? Justify your answer.

[*15 marks*]

Answer 4.

f has exactly one fixed point, since it is order-reversing (recall that all order-reversing diffeomorphisms have precisely one fixed point).

To see that f is order-reversing, note that $f'(x) = -1 - \frac{1}{2}\sin x \leq -1/2 < 0$ for all $x \in \mathbb{R}$.

Answer 4. (*Continue*)

Question 5.

Determine those natural numbers n such that every continuous function $f : \mathbb{R} \to \mathbb{R}$ with an n-cycle also has an n^2-cycle.

[*15 marks*]

Answer 5.

The set of such numbers is

$$\mathbb{N} \setminus \{2^k : k \geq 1\} .$$

To see this, note first that if $n = 1$ then $n^2 = 1$, so clearly every continuous function $f : \mathbb{R} \to \mathbb{R}$ with a 1-cycle also has a 1-cycle.

Next note that if n has some factor apart from 2 (i.e. $n \neq 2^k$ for $k \geq 1$) then n^2 is *smaller* than n in the Sharkovskii order, so Sharkovskii's Theorem tells us that every continuous function $f : \mathbb{R} \to \mathbb{R}$ with an n-cycle also has an n^2-cycle.

Finally note that if $n = 2^k$ for some $k \geq 1$ then n^2 is *larger* than n in the Sharkovskii order, so it is not the case that every continuous function $f : \mathbb{R} \to \mathbb{R}$ with an n-cycle also has an n^2-cycle.

Answer 5. (*Continue*)

Question 6.

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$ and $g(x) = -x^3$. Are f and g topologically conjugate to each other? Justify your answer.

[*15 marks*]

Answer 6.

No, f and g are *not* topologically conjugate.

To see this, note that f has 3 fixed points and g only has one fixed point, whereas if they were topologically conjugate then they would have the same number of fixed points.

The 3 fixed points of f are 0, 1, and -1 (i.e. the three roots of $f(x)-x = x^3-x = x(x-1)(x+1)$. The single fixed point of g is 0 (note that $g(x) - x = -x^3 - x = -x(x^2 + 1)$ only has one real root).

Answer 6. (*Continue*)