Lecture 2

Theorem 2: Let $GCD(s,t) = 1$ s.t.

$$\left|r - \frac{s}{t}\right| < \frac{1}{2t^2}.$$ Then, $\frac{s}{t}$ is a convergent to r.

Proof: Choose n s.t. $t_{n-1} \leq t < t_n$

Prop 2: $$|tr - s| \geq |t_{n-1} r - s_{n-1}|$$

$$t\left|r - \frac{s}{t}\right| < \frac{t}{2t^2} = \frac{1}{2t}$$

$$\Rightarrow \left|r - \frac{s_{n-1}}{t_{n-1}}\right| \leq \frac{1}{2t_{n-1}t}.$$

We see, i) $$\left|\frac{s}{t} - \frac{s_{n-1}}{t_{n-1}}\right| \leq \left|\frac{s}{t} - r\right| + \left|r - \frac{s_{n-1}}{t_{n-1}}\right|$$

$$\leq \frac{1}{2t^2} + \frac{1}{2t_{n-1}t} \leq \frac{1}{t_{n-1}t}$$

as $t \geq t_{n-1}$

ii) $$\left|\frac{s}{t} - \frac{s_{n-1}}{t_{n-1}}\right| = \left|\frac{s t_{n-1} - t s_{n-1}}{t t_{n-1}}\right|$$

2) $$|s t_{n-1} - t s_{n-1}| < 1 \Rightarrow s t_{n-1} = t s_{n-1}$$

$$\Rightarrow \frac{s}{t} = \frac{s_{n-1}}{t_{n-1}}.$$

Periodic continued fraction revisited

Def: A quadratic irrational is a real number $r \in \mathbb{R}$ of the form $r = s + t\sqrt{d}$ where

- $s \in \mathbb{Q}$
- $t \in \mathbb{Q} \setminus \{0\}$
- $d \in \mathbb{N}$, d is square-free

$$\nexists \; p \text{ prime s.t. } p^2 \mid d,$$

in other words, prime factorization of $d = p_1 p_2 \cdots p_k$ where p_i are distinct primes.

Theorem 3: If a real numbers has a periodic continued fraction, then it is a quadratic irrational.

RMK: The converse is also true. That is, every quadratic irrational has a periodic continued fraction expansion.

<u>Proof</u>: Let the continued fraction be purely periodic, i.e.

$$r = [\overline{a; a_1, \ldots, a_\ell}].$$

[<u>Exercise</u>: Directly check if $\ell = 0$ or $\ell = 1$ then $r \in \mathbb{Q} + \mathbb{Q}\sqrt{d}$]

We assume that $\ell > 1$.

From the lemma from morning we have $r = [a; a_1, \ldots, a_{\ell-1}, r]$

$$= \frac{r s_\ell + s_{\ell-1}}{r t_\ell + t_{\ell-1}}$$

where $\frac{s_\ell}{t_\ell} = r_\ell$ is the ℓ'th convergent to r.

Thus $r^2 t_\ell + r(t_{\ell-1} - s_\ell) - s_{\ell-1} = 0.$

Solution of this is of the form $\mathbb{Q} + \mathbb{Q}^\times\sqrt{d}$. (convince yourself why $\mathbb{Q}^\times$!)

Now assume that, $r = [a, a_1 .., a_N, \overline{a_{N+1}, \ldots, a_{N+\ell}}]$

We know $r = [a; a_1, \ldots, a_N, r_1]$

where $r_1 \in \mathbb{Q} + \mathbb{Q}^{\times}\sqrt{d}$.

Once again using the lemma from morning we have

$$r = \frac{r_1 s_N + s_{N-1}}{r_1 t_N + t_{N-1}}$$

If, $r_1 = s + t\sqrt{d}$ then

$$r = \frac{s s_N + s_{N-1} + t\sqrt{d}\, s_N}{s t_N + t_{N-1} + t\sqrt{d}\, t_N}$$

is of the form $\dfrac{a + b\sqrt{d}}{c + e\sqrt{d}}$, $a, b, c, d \in \mathbb{Z}$

But the above $= \dfrac{(a + b\sqrt{d})(c - e\sqrt{d})}{c^2 - e^2 d}$

$$= \frac{ac - bed}{c^2 - e^2 d} + \frac{(bc - ae)}{e^2 - e^2 d}\sqrt{d}$$

$$\in \mathbb{Q} + \mathbb{Q}\sqrt{d}$$

$$r = \mathbb{Q} + \frac{s_{N-1}\, t\, (s_N - t_N)}{(t_N s + t_{N-1})^2 - (t_N t)^2 d}\sqrt{d}$$

As $s_N \neq t_N$ (as $r_N \neq 1$)

we conclude.

Exercise: $r \in \mathbb{R} \setminus \mathbb{Q}$ with continued fraction expansion $r = [a; a_1, \ldots]$.

Let $r_n = \frac{s_n}{t_n} = [a; a_1, \ldots, a_n]$ and $s, t \in \mathbb{N}$ with $t < t_n$.

1) Solve:

$$s_{n-1} x + s_n y = s$$
$$t_{n-1} x + t_n y = t.$$

[The solution $(x, y) = \left((-1)^n (s t_n - t s_n), (-1)^n (t s_{n-1} - s t_{n-1})\right)$ is unique.]

2) Show ~~that~~ $x \neq 0$.

3) Show that $y \neq 0$ or $tr - s = t_{n-1} r - s_{n-1}$.

Lemma: Let $y \neq 0$. Then x & y must have different signs.

Proof: Let $y < 0$ and n be odd.

Then $\quad -(t s_{n-1} - s t_{n-1}) < 0$

$\Rightarrow \quad \frac{s}{t} < \frac{s_{n-1}}{t_{n-1}} = r_{n-1}$

But as n is odd $\quad \frac{s}{t} < r_{n-1} < r_n = \frac{s_n}{t_n}$

$\Rightarrow \quad s t_n - t s_n < 0$

Thus $\quad x = (-1)^n (s t_n - t s_n) > 0$

as n is odd.

<u>Exercise</u>: check the cases

i) $y < 0$ & n even
ii) $y > 0$ & n odd
iii) $y > 0$ & n even.

<u>Exercise</u>: $t_{n-1} r - s_{n-1}$ & $t_n r - s_n$ must have opposite signs.

<u>Proof</u>: If n even then

$$r_n = \frac{s_n}{t_n} < r < \frac{s_{n-1}}{t_{n-1}} = r_{n-1}$$

$\Rightarrow \quad r t_n - s_n > 0 \quad$ & $\quad r t_{n-1} - s_{n-1} < 0$.

Completion of the proof of Prop 2

Combining last two facts we have

$$x(t_{n-1} r - s_{n-1}) \ \& \ y(t_n r - s_n)$$

must have the same signs.

Thus $|tr - s|$

$$= |(t_{n-1} x + t_n y) r - (s_{n-1} x + s_n y)|$$

$$= |x(t_{n-1} r - s_{n-1}) + y(t_n r - s_n)|$$

$$\overset{\text{Same sign}}{=} |x|\,|t_{n-1} r - s_{n-1}| + |y|\,|t_n r - s_n|$$

$$> |t_{n-1} r - s_{n-1}|$$

As $|x| \geq 1$, $y \neq 0$ and

$t_n r - s_n \neq 0$

as $r \neq \frac{s_n}{t_n} \in \mathbb{Q}$.