Lecture 1

<u>Recall</u>, $r = [a; a_1, a_2, \ldots]$ an irrational number. Its n'th convergent is given by $[a; a_1, \ldots a_n]$ which we denote by $r_n \in \mathbb{Q}$.

<u>Goal</u>: Determine the best possible approximation $\frac{s}{t}$ to r.

1) It should be very close to r
2) The denominator t needs to be small.

<u>Further recall</u>

1) $r_n = \dfrac{s_n}{t_n}$, $\quad s_{n+1} = a_n s_{n-1} + s_{n-2}$
$\qquad\qquad\qquad\quad t_{n+1} = a_n t_{n-1} + t_{n-2}$

2) $r_0 < r_2 < r_4 < \cdots < r < \cdots < r_5 < r_3 < r_1$

3) $\left.\begin{array}{l} \text{If } n \text{ is even then} \\ \quad r_n < r < r_{n+1} \\ \text{If } n \text{ is odd then} \\ \quad r_{n+1} < r < r_n \end{array}\right\}$ Exercise.

4) $|r_{n+1} - r_n| = \frac{1}{t_n t_{n+1}}$

Thus $|r - r_n| < \frac{1}{t_n \cdot t_{n+1}}$.

<u>Example</u>: $r = \sqrt{15} - 3$

$= [0; 1, 6, 1, 6 \ldots] = [0; \overline{1, 6}]$

$r_0 = 0$, $r_1 = 1$, $r_2 = \frac{6}{7}$, $r_3 = \frac{7}{8}$

$r_4 = \frac{48}{55}$, $r_5 = \frac{55}{63}$, $r_6 = \frac{378}{433}$

Note that $|r - r_6| = \left|\sqrt{15} - 3 - \frac{378}{433}\right|$

$< \frac{1}{433 \times 496} < 10^{-4}$

i.e. approximate upto 4-decimal place.

<u>RMK</u>: r_n is always better approximant than r_{n-2}. But how do we compare r_n vs. r_{n+1} ?

We will show that approximation always gets better as n increases.

Recall the algorithm for continued fraction.

$a = \lfloor r \rfloor$, $\quad \rho_1 = \dfrac{1}{r-a} \Rightarrow a_1 = \lfloor \rho_1 \rfloor$

$\rho_2 = \dfrac{1}{\rho_1 - a_1} \Rightarrow a_2 = \lfloor \rho_2 \rfloor$

$\vdots$

Thus $\rho_1 = [a_1; a_2, \dots]$

$\rho_n = [a_n; a_{n+1}, \dots]$

On the other hand, by definition

$$r = [a; a_1, \dots, a_n, \rho_{n+1}]$$

<u>Lemma</u>: $$r = \frac{\rho_{n+1} s_n + s_{n-1}}{\rho_{n+1} t_n + t_{n-1}}$$

<u>Proof</u>: Induction on n.

$n = 0$: Need to show $r = \dfrac{\rho_1 s_0 + s_{-1}}{\rho_1 t_0 + t_{-1}}$

$$= \frac{\rho_1 a + 1}{\rho_1 + 0} = a + \frac{1}{\rho_1} \quad ✓$$

But $\rho_1 = \dfrac{1}{r-a}$.

Now assume that the identity holds for $n-1$.

i.e. $\quad r = [a; a_1, \dots a_{n-1}, \rho_n]$

$$= \frac{\rho_n s_{n-1} + s_{n-2}}{\rho_n t_{n-1} + t_{n-2}}$$

But $\quad [a; a_1 \dots a_n, \rho_{n+1}]$

$$= \left[a; a_1, \dots, a_{n-1}, a_n + \frac{1}{\rho_{n+1}}\right]$$

$$= \frac{\left(a_n + \frac{1}{\rho_{n+1}}\right) s_{n-1} + s_{n-2}}{\left(a_n + \frac{1}{\rho_{n+1}}\right) t_{n-1} + t_{n-2}}$$

$$= \frac{(a_n s_{n-1} + s_{n-2}) \rho_{n+1} + s_{n-1}}{(a_n t_{n-1} + t_{n-2}) \rho_{n+1} + t_{n-1}}$$

$$= \frac{s_n \rho_{n+1} + s_{n-1}}{t_n \rho_{n+1} + t_{n-1}} \qquad \square$$

Thus, $\quad r - \frac{s_n}{t_n}$

$$\frac{\rho_{n+1} s_n + s_{n-1}}{\rho_{n+1} t_n + t_{n-1}} - \frac{s_n}{t_n} = \frac{t_n s_{n-1} - t_{n-1} s_n}{t_n(\rho_{n+1} t_n + t_{n-1})}$$

$$= \frac{(-1)^n}{t_n(\rho_{n+1} t_n + t_{n-1})}$$

Now $a_{n+1} = \lfloor \rho_{n+1} \rfloor$

$\Rightarrow a_{n+1} < \rho_{n+1}$

thus $\left| r - \frac{s_n}{t_n} \right| < \frac{1}{t_n (a_{n+1} t_n + t_{n-1})}$

$= \frac{1}{t_n t_{n+1}} \to 0.$

Prop 1 For $n \geq 2$ we have

1) $| t_n r - s_n | < | t_{n-1} r - s_{n-1} |$

2) $| r - r_n | < | r - r_{n-1} |$

Proof : $| t_n r - s_n | = t_n \left| r - \frac{s_n}{t_n} \right|$

$= \frac{1}{\rho_{n+1} t_n + t_{n-1}}$ using above relation.

Similarly, $| t_{n-1} r - s_{n-1} | = \frac{1}{\rho_n t_{n-1} + t_{n-2}}$

1) $\Longleftrightarrow$ $\rho_{n+1} t_n + t_{n-1} > \rho_n t_{n-1} + t_{n-2}$

As $\rho_{n+1} = \frac{1}{\rho_n - \lfloor \rho_n \rfloor} > 1$ and

$\rho_n < a_n + 1$

Thus, $LHS > t_n + t_{n-1}$

$$= a_n t_{n-1} + t_{n-2} + t_{n-1}$$

$$= t_{n-1}(a_n + 1) + t_{n-2}$$

$$> t_{n-1} \rho_n + t_{n-2} \quad = RHS.$$

To prove 2) we see,

$$|r - r_n| = \left|r - \frac{s_n}{t_n}\right| = \left|\frac{t_n r - s_n}{t_n}\right|$$

$$\overset{(1)}{<} \left|\frac{t_{n-1} r - s_{n-1}}{t_n}\right| < \left|\frac{t_{n-1} r - s_{n-1}}{t_{n-1}}\right|$$

$$= |r - r_{n-1}|$$

as $t_n > t_{n-1}$. $\boxtimes$

<u>Def</u>: We say that a rational number $\frac{s}{t}$ is a <u>good approximation</u> to r if $\left|r - \frac{s}{t}\right| < \left|r - \frac{s'}{t'}\right|$ $\forall \frac{s'}{t'} \in \mathbb{Q}$ with $t' < t$.

Theorem 1: The convergents are good approximants, i.e. we have

$$\left| r - \frac{s_n}{t_n} \right| < \left| r - \frac{s}{t} \right| \quad \forall \; t < t_n$$

$\left[\frac{s}{t}\right.$ is in its lowest terms$\left.\right]$

Prop 2: $r_n = [a; a_1, \ldots, a_n] = \frac{s_n}{t_n}$. We have for any $GCD(s,t) = 1$, $t < t_n$

$$|tr - s| \geq |t_{n-1} r - s_{n-1}|.$$

with equality holding iff

$$\frac{s}{t} = \frac{s_{n-1}}{t_{n-1}}.$$

Proof of the Theorem assuming the Prop

Indeed, $\left| r - \frac{s}{t} \right| = \left| \frac{tr - s}{t} \right|$

$$\overset{\text{Prop 2}}{\geq} \frac{|t_{n-1} r - s_{n-1}|}{t} \overset{\text{Prop 1}}{\geq} \frac{|t_n r - s_n|}{t} \overset{t < t_n}{\geq} \frac{|t_n r - s_n|}{t_n} = \left| r - \frac{s_n}{t_n} \right|. \quad \boxtimes$$