PROBLEM SET 9 FOR MTH 6151

1. Show that if $U(x,t)$ is a solution to the heat equation then $U(\alpha x, \alpha^2 t)$, α a constant, is also a solution to the heat equation. What about $U(\alpha x, -\alpha^2 t)$, is this a solution too?

2. Find all the values of the constants a and b such that

$$U(x,t) = e^{ax+bt}$$

satisfies the heat equation.

3. Use the method of separation of variables to solve the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \quad x \in [0, \frac{\pi}{2}], \quad t \geq 0, \\ &U(x,0) = \sin x, \\ &U(0,t) = 0, \qquad U_x(\frac{\pi}{2}, t) = 0. \end{aligned}$$

Plot the solutions at several instants of time and describe in qualitative terms the behaviour of the solution to as $t \to \infty$. What is $\lim_{t\to\infty} U(x,t)$?

(Observe that this problem has mixed boundary conditions —Dirichlet to the left and Neumann to the right.)

4. Use the method of separation of variables to solve the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \quad x \in [0,1], \quad t \geq 0, \\ &U(x,0) = -6\sin(6\pi x), \\ &U(0,t) = 0, \qquad U(1,t) = 0. \end{aligned}$$

Plot the solutions at several instants of time and describe in qualitative terms the behaviour of the solution to as $t \to \infty$. What is $\lim_{t\to\infty} U(x,t)$?

5. Use the method of separation of variables to solve the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \quad x \in [0,L], \quad t \geq 0, \\ &U(x,0) = f(x), \\ &U_x(0,t) = 0, \qquad U_x(L,t) = 0. \end{aligned}$$

Observe that this problem has Neumann boundary conditions.

6. Compute the solution of Exercise 5 if:

(i) $f(x) = 1$,
(ii) $f(x) = \cos^2(\pi x/L)$.

Plot the solutions at several instants of time and describe in qualitative terms the behaviour of the solution to as $t \to \infty$. What is $\lim_{t\to\infty} U(x,t)$?

7. Use the method of separation of variables to solve the problem

$$\begin{aligned} &U_t = \varkappa U_{xx}, \quad x \in [-L, L], \quad t \geq 0, \\ &U(x, 0) = f(x), \qquad f(L) = f(-L). \\ &U(-L, t) = U(L, t), \\ &U_x(-L, t) = U_x(L, t). \end{aligned}$$

The boundary conditions used in this problem are called periodic. Can you imagine a physical system described by this problem?

8. Compute the solution in Exercise 7 in the case

$$f(x) = \sin^2(\pi x/L).$$

What is the value of $U(x, t)$ as $t \to \infty$?