Machine Learning with Python MTH786U/P 2023/24

Week 8: Classification tasks

Nicola Perra, Queen Mary University of London (QMUL)

Similar to regression, *classification* relates input variables $\{\mathbf{x}_i\}_{i=1}^s$ to output variables $\{\mathbf{y}_i\}_{i=1}^s$, i.e.

$$\mathbf{y}_i \approx f(\mathbf{x}_i) \quad \forall i \in \{1, ..., s\}$$



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Key-difference in classification:



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Example:
$$\mathbf{y} \in \{-1,0,1\}^s$$
 with an example vector

$$y = (0, -1, -1, 1, 0)^T$$
 for $s = 5$

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Note that even if the class labels take on numerical values, there is typically no ordering implied between the two classes

Binary classification $oldsymbol{x}_i \in \mathbb{R}^d$ TO TIS !!!! Training data image source

Examples:

Examples: Surviving the titanic disaster



©Wikimedia commons

Either you have survived or not survived the sinking of the Titanic (assuming you were a passenger on the Titanic)

Examples: Train delays

Often formulated and treated as a regression problem

©Evening Standard





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Often formulated and treated as a regression problem

Can also be considered a binary classification problem:

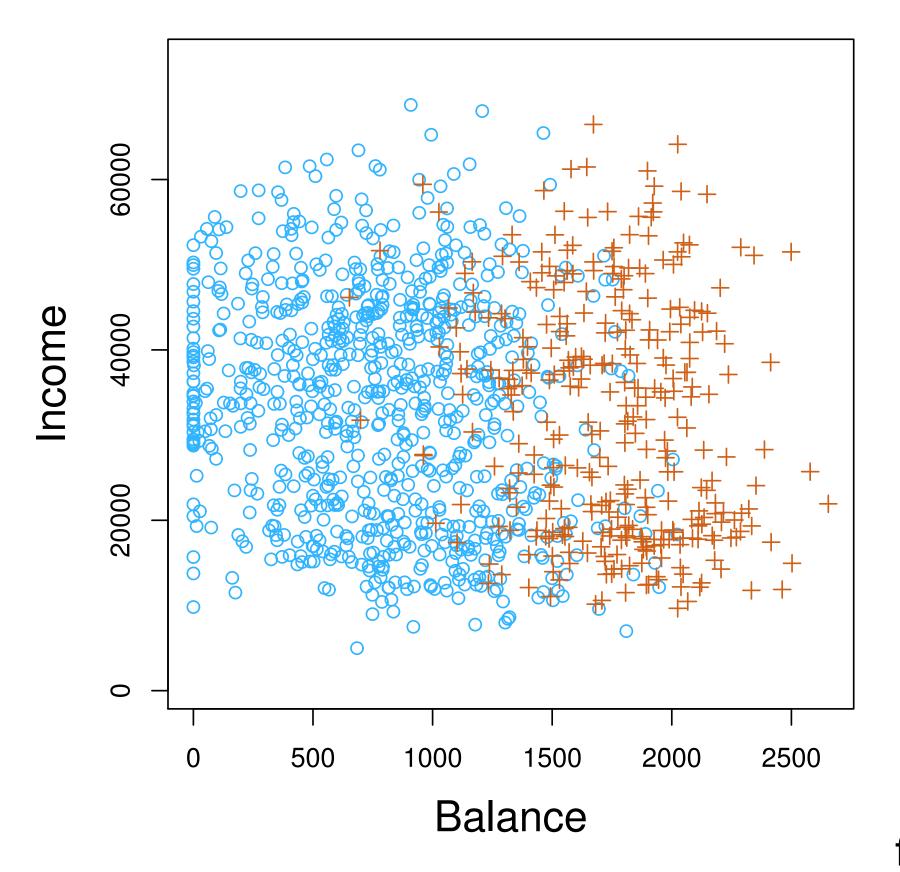






$$y_i = \begin{cases} C_1 & \text{train delay} < 30 \text{ minutes (no refund)} \\ C_2 & \text{train delay} \ge 30 \text{ minutes (refund)} \end{cases}$$

Examples: credit default

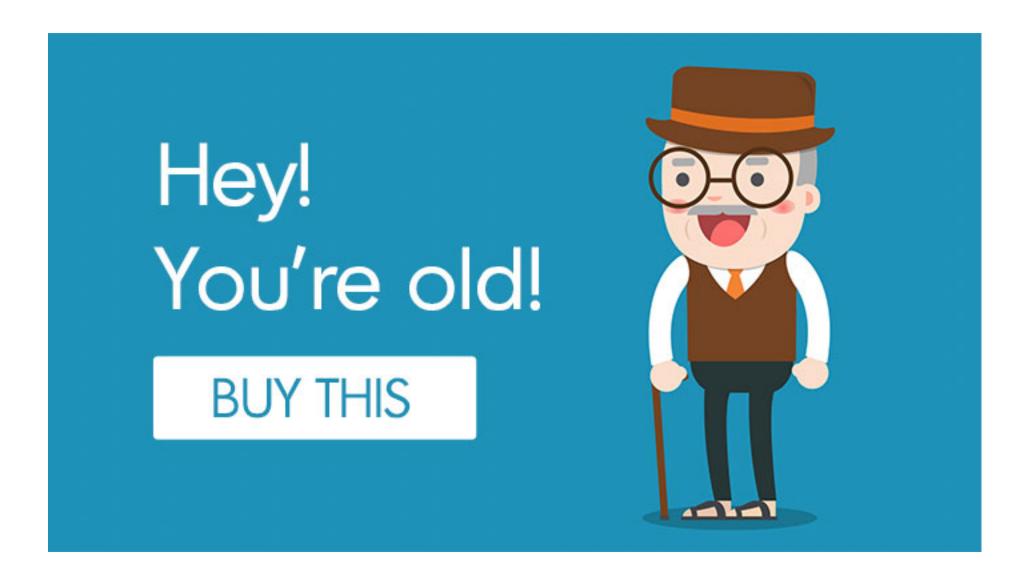


- + = individual who defaulted on their credit card payments
- o = individual who did not default on their credit card payments

from *Elements of Statistical Learning* by Hastie, Tibshirani and Friedman

Examples: Targeted marketing

Classify customers as likely buyers vs unlikely buyers of product X





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Classify customers as likely buyers vs unlikely buyers of product X



Image source

Be aware of ethical consequences, e.g. voter targeting



In multi-class classification, y can take on more than two values, i.e.

$$y \in \{C_0, C_1, ..., C_{K-1}\}^s$$

for the K class labels $C_0, ..., C_{K-1}$



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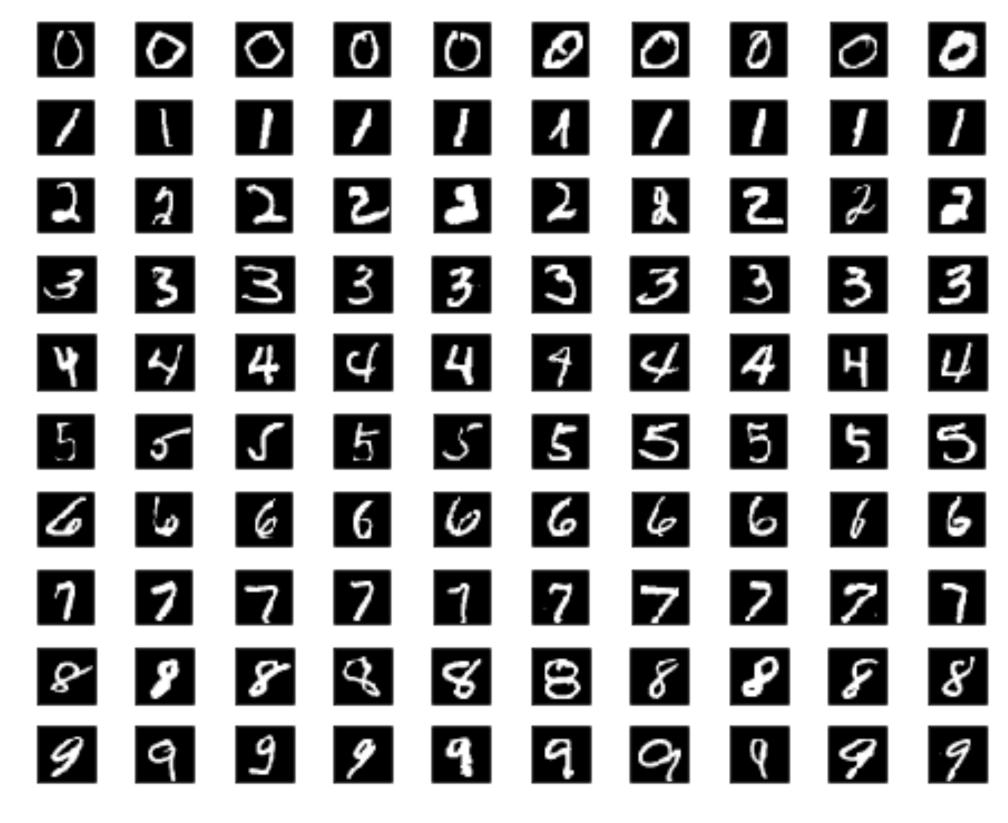
$$y \in \{C_0, C_1, ..., C_{K-1}\}^s$$

for the K class labels $C_0, ..., C_{K-1}$

Again, there is in general no ordering amongst the classes but often we will use numerical values as class labels, e.g.

$$y \in \{0,1,...,K-1\}^s$$

Example: classification of hand-written digits



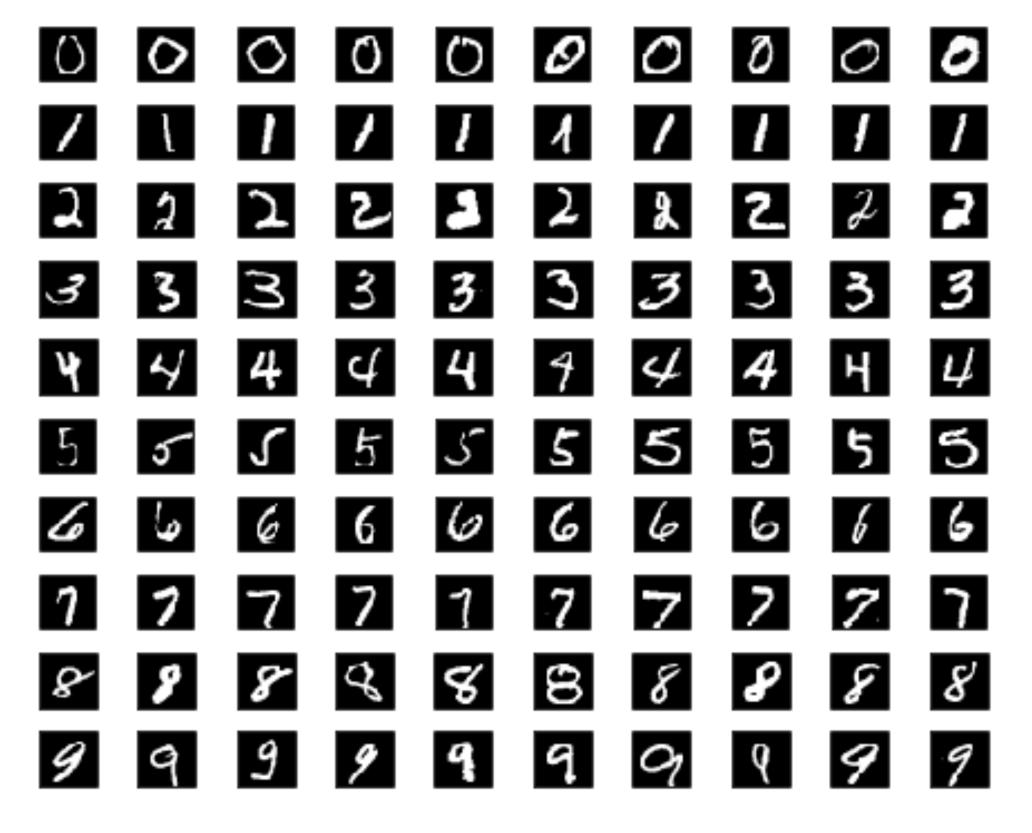


MNIST database

Example: classification of hand-written digits

Decide whether an image of a handwritten digit belongs to class

0,1,2,3,4,5,6,7,8 or 9







In order to classify, we need a *classifier*



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The boundaries of these regions are called decision boundaries



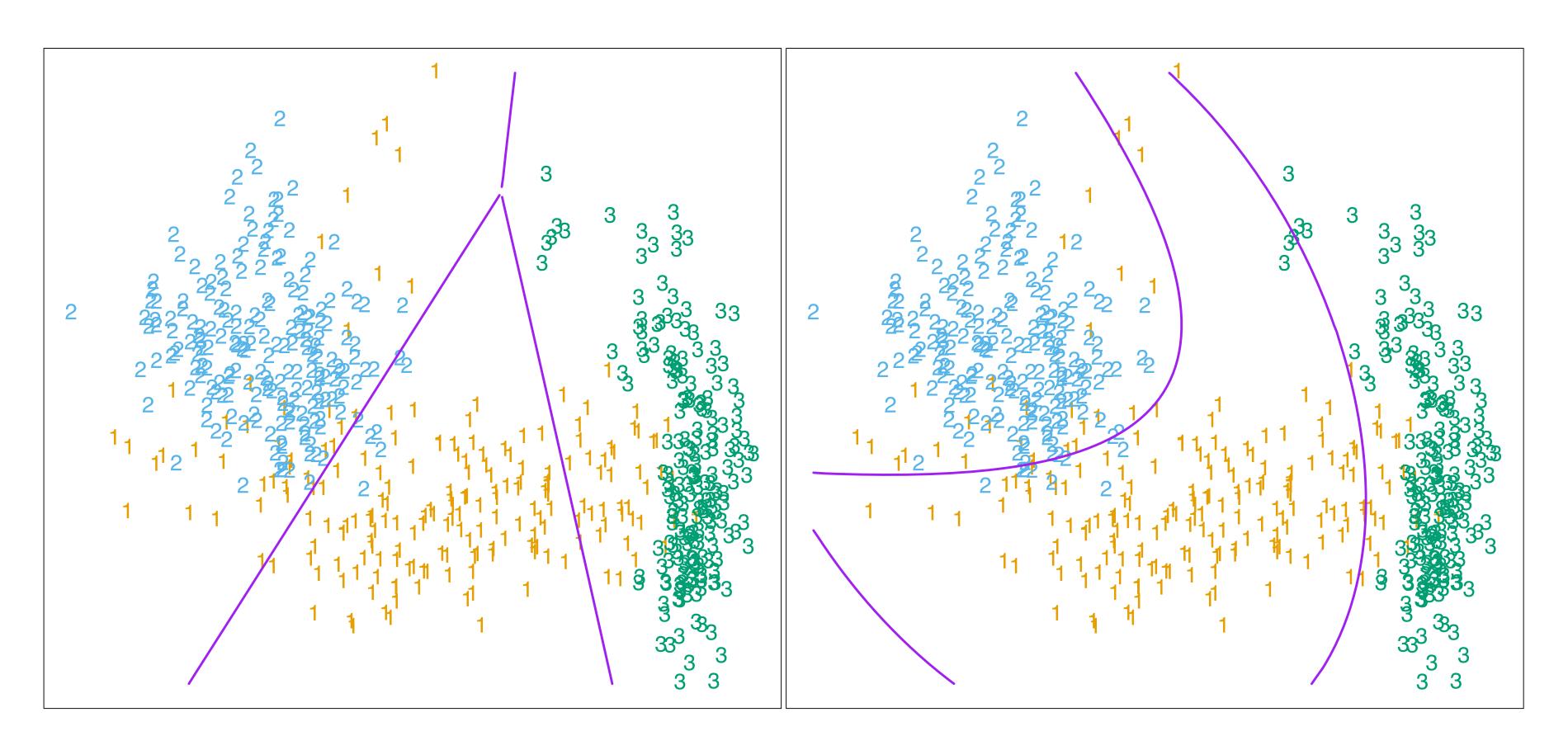
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We distinguish between linear and nonlinear classifiers





from *Elements of Statistical Learning* by Hastie, Tibshirani and Friedman

Classification itself: we are constructing a predictor based on a training set and are interested in applying the predictor to new data



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For the second task it is often important to have 'simple' models

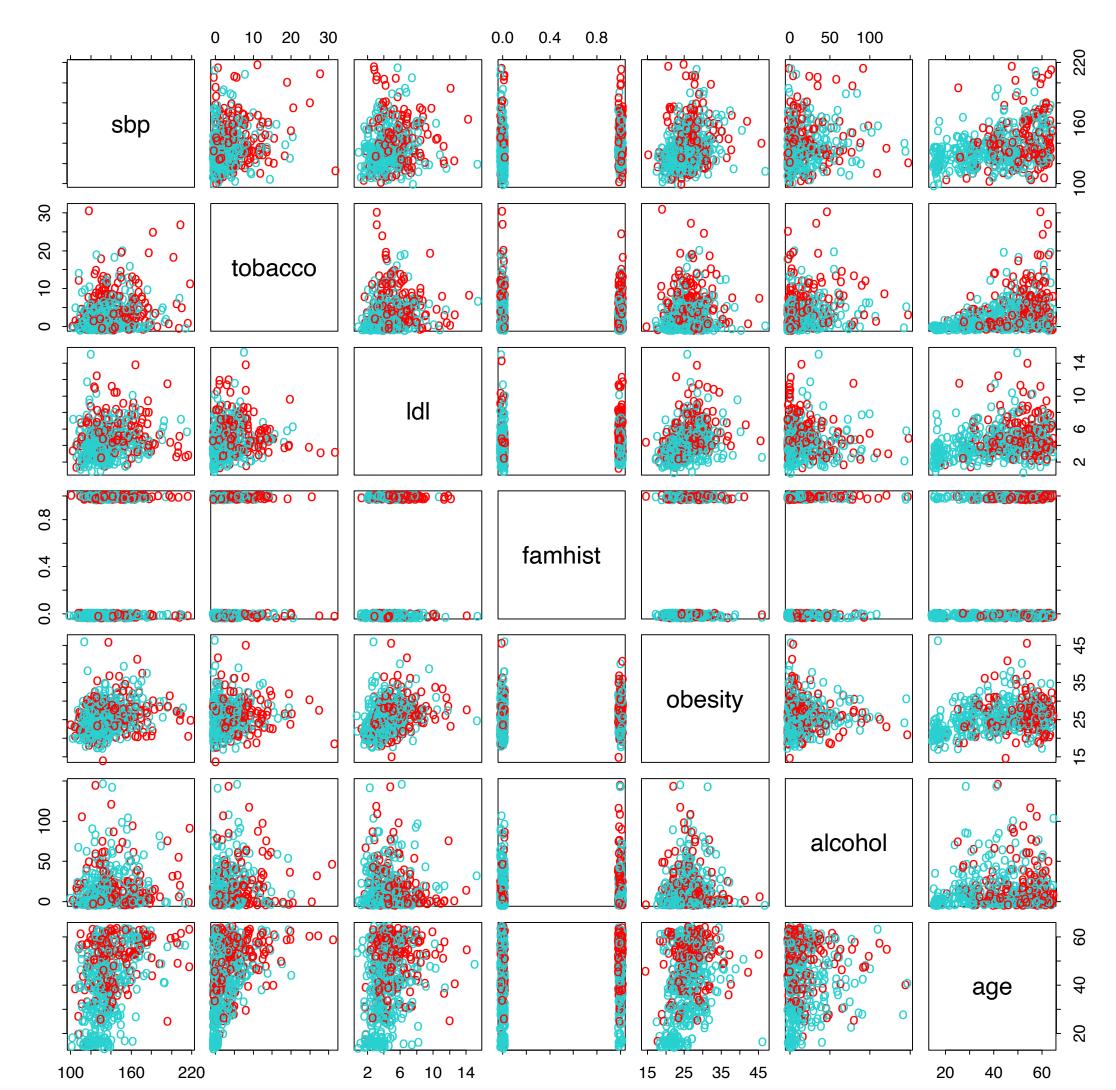
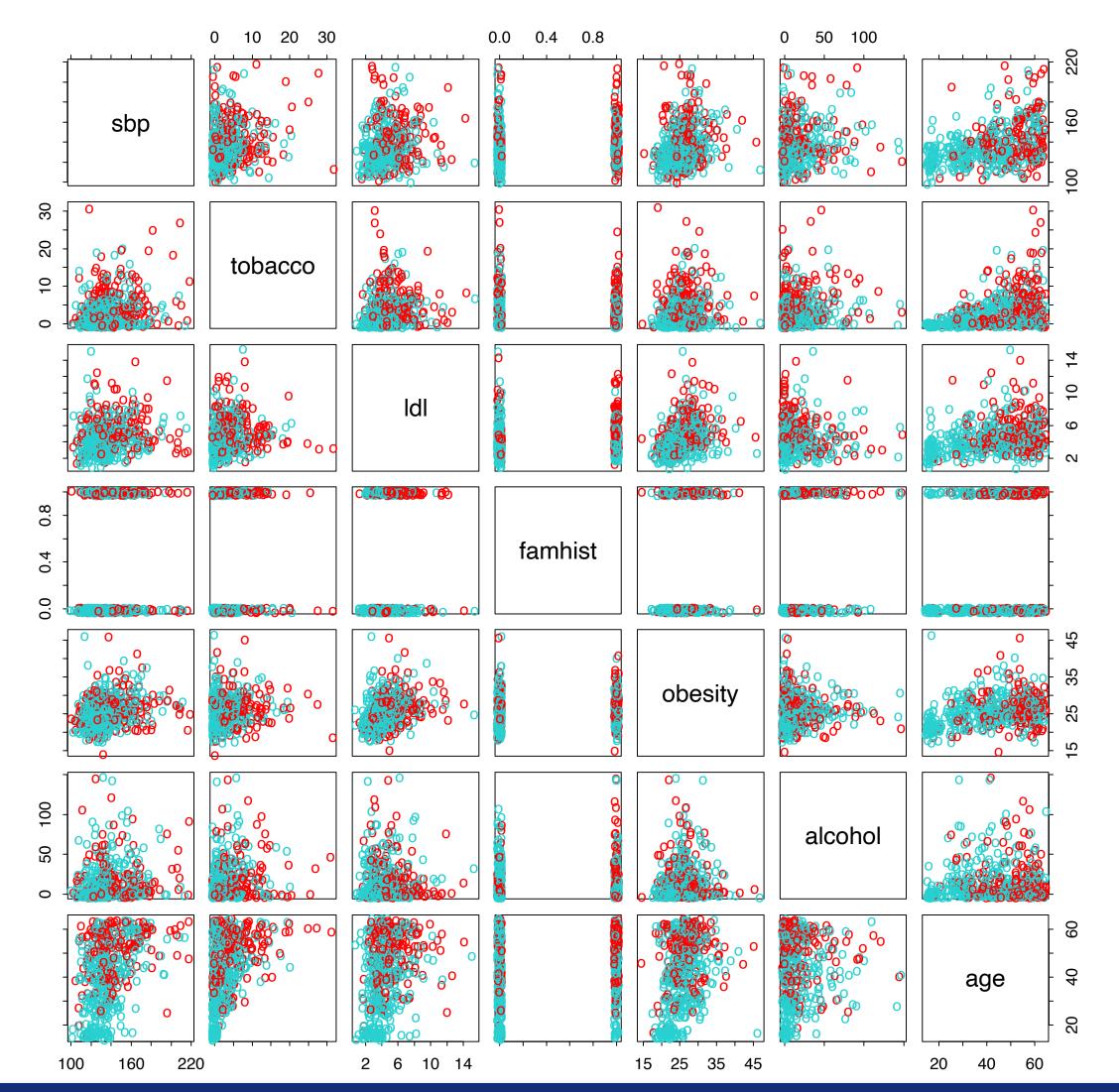


FIGURE 4.12. A scatterplot matrix of the South African heart disease data. Each plot shows a pair of risk factors, and the cases and controls are color coded (red is a case). The variable family history of heart disease (famhist) is binary (yes or no).

from *Elements of Statistical Learning* by Hastie, Tibshirani and Friedman





Scatterplots like this one can help to decide which risk factors should be included in a model

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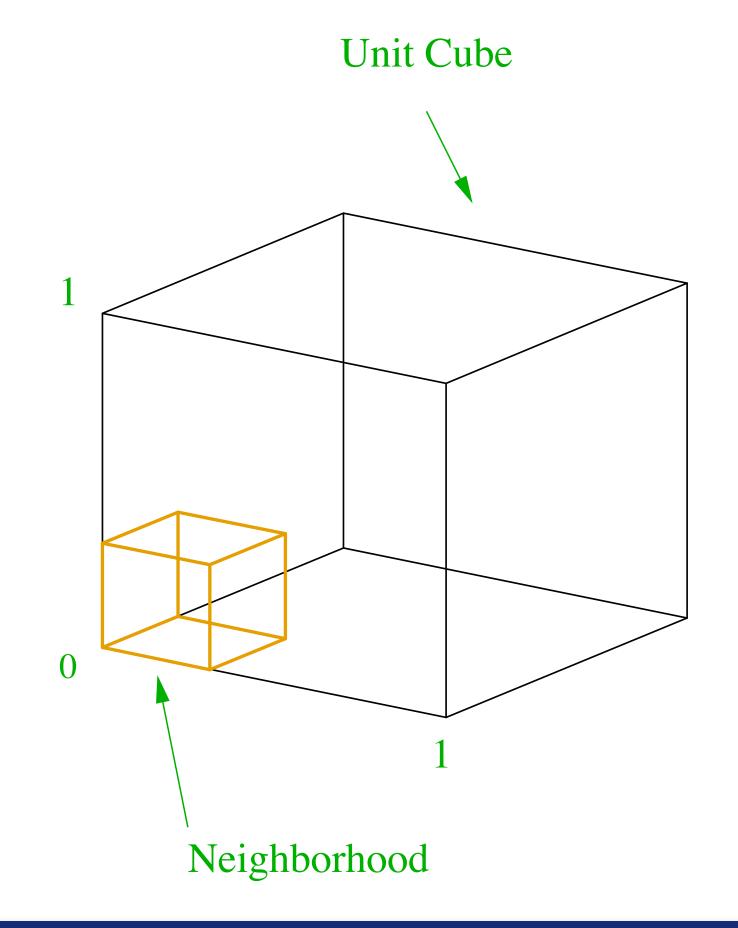
Imagine all points lie in d-dimensional unit cube $[0,1]^d$



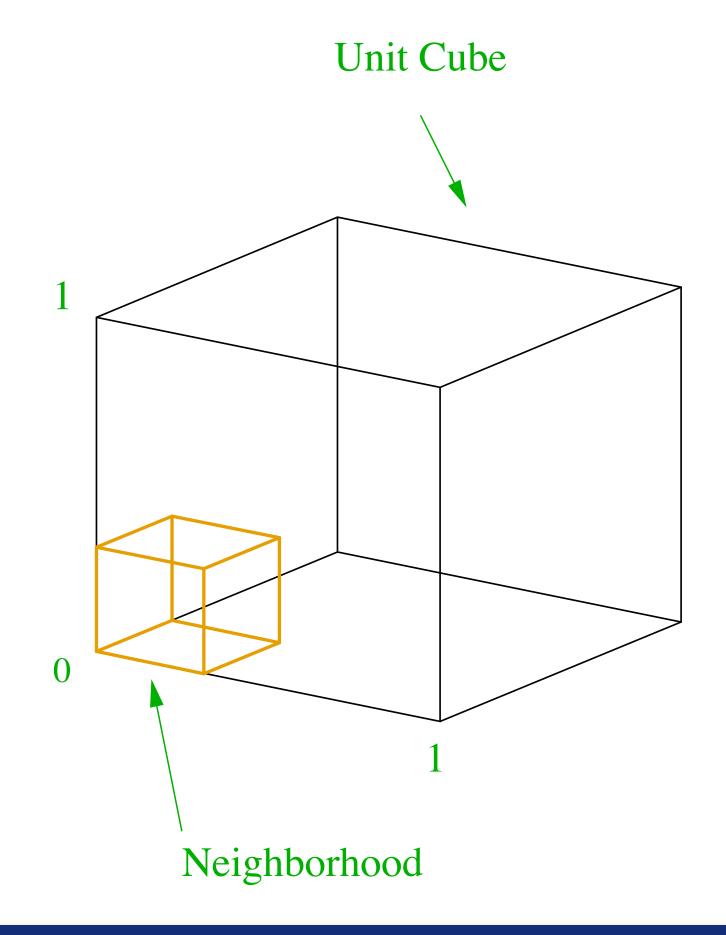
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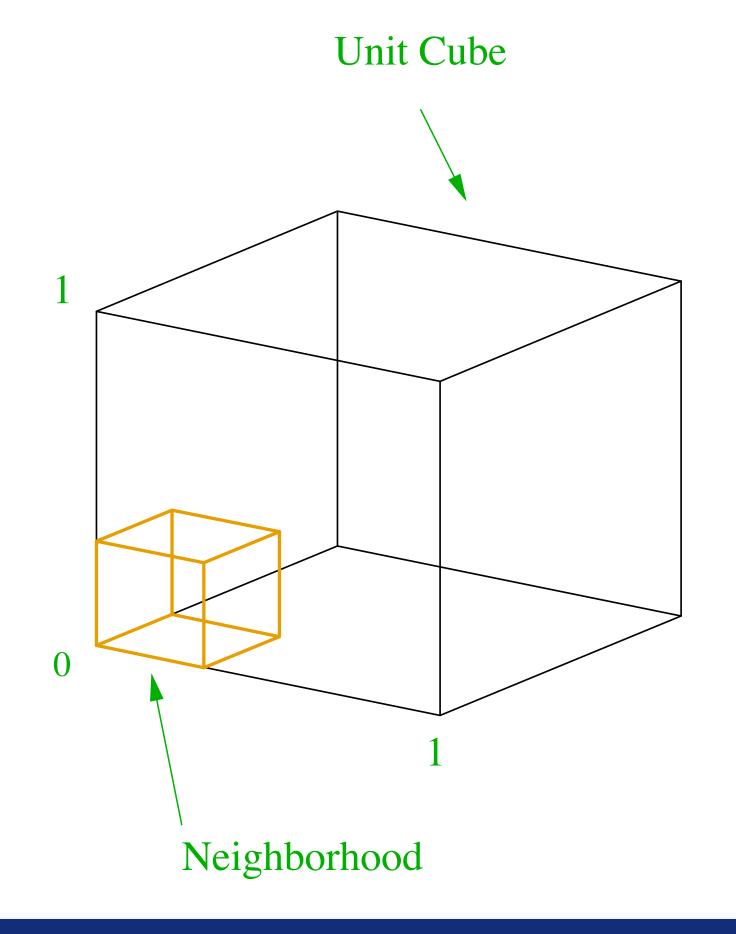


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Consider sub-cube $[a, a+r]^d \subset [0,1]^d$ with $0 \le a$ and $a+r \le 1$

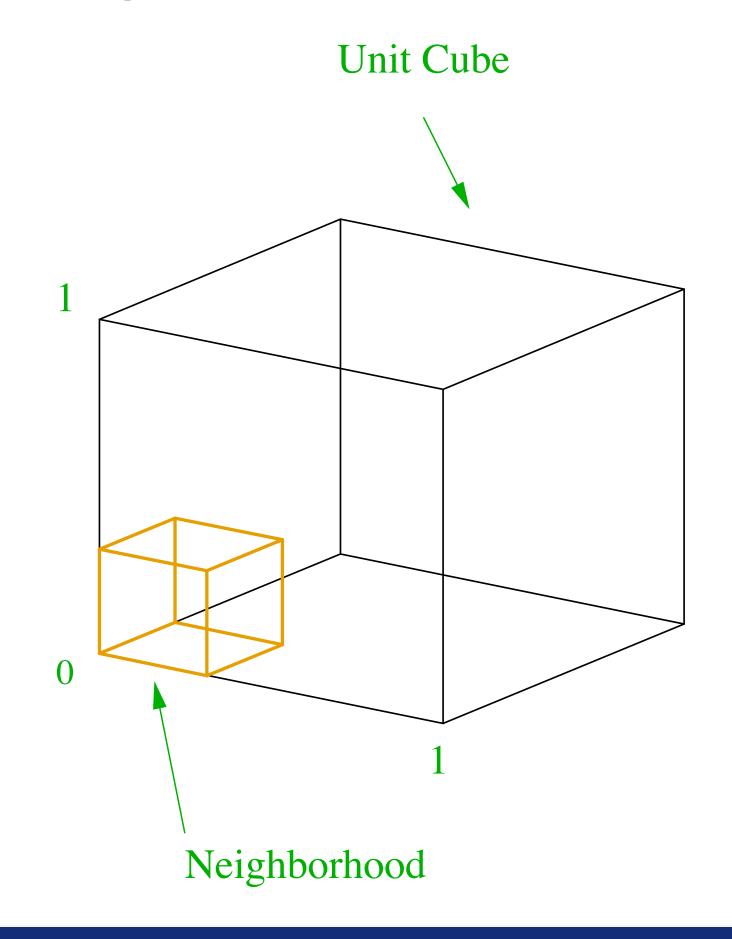
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What fraction of the total volume does this cube cover?

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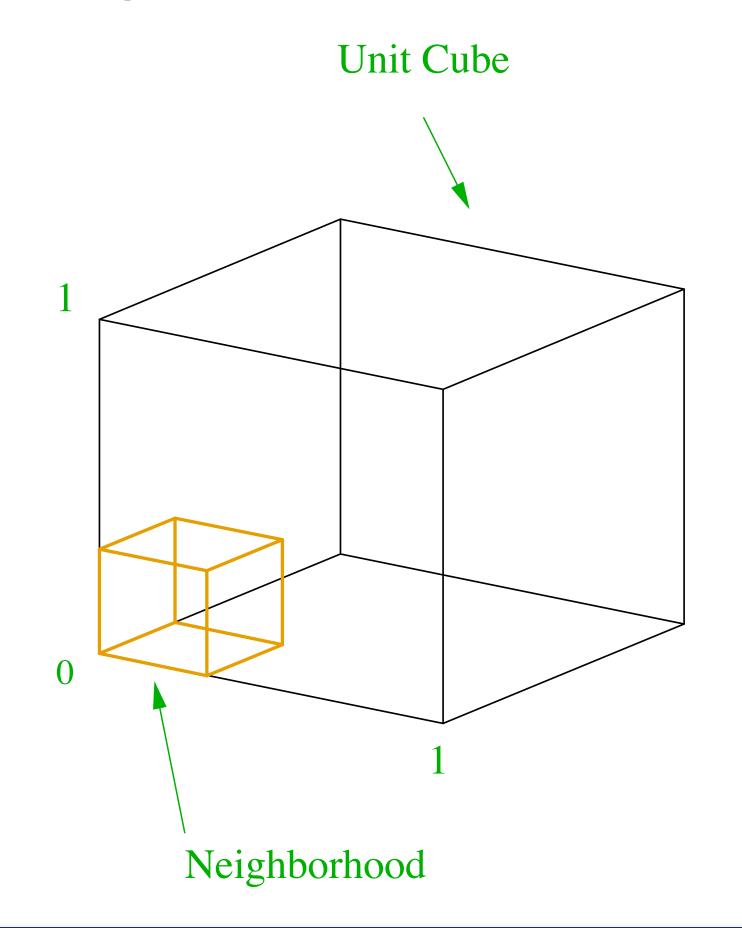


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Answer: r^d

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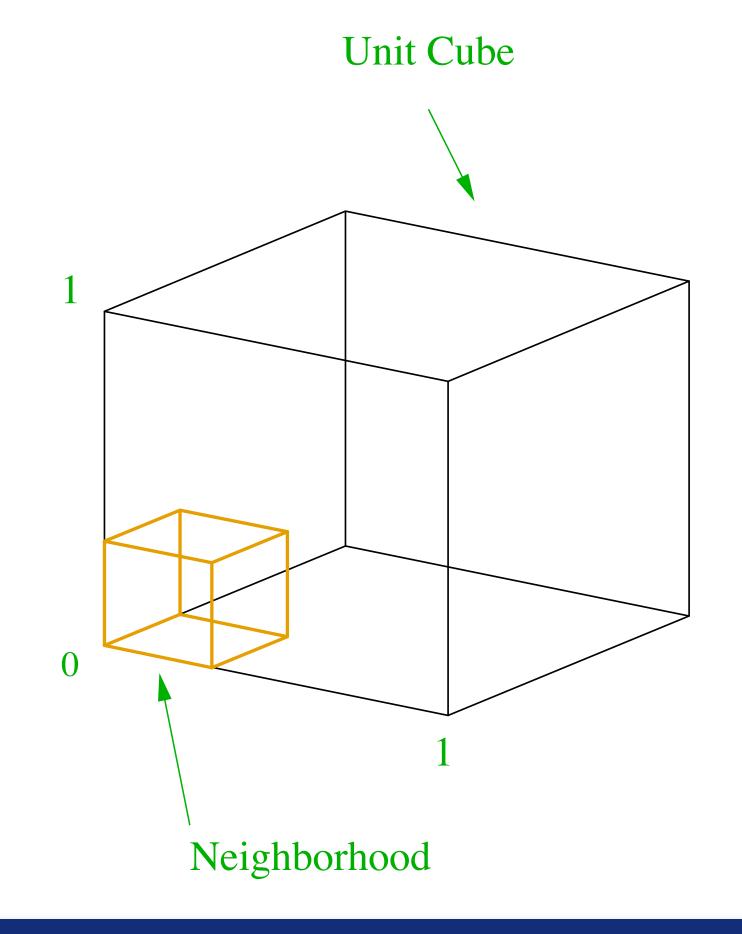


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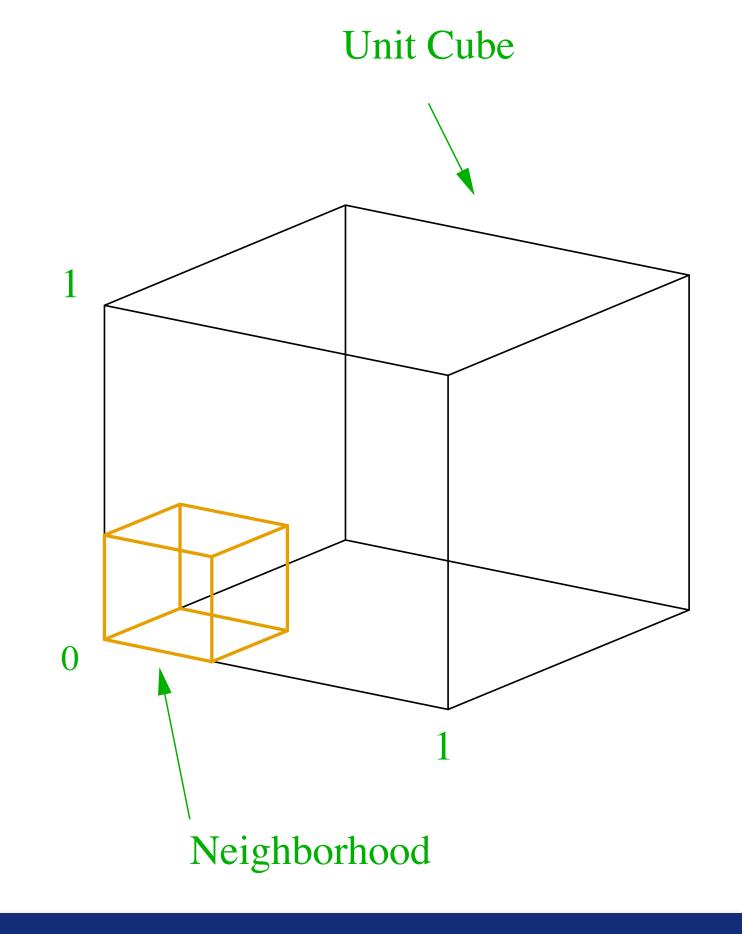
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$$\begin{array}{c} \alpha = 1 \% \\ d = 5 \end{array} \Rightarrow r \approx 0.4$$

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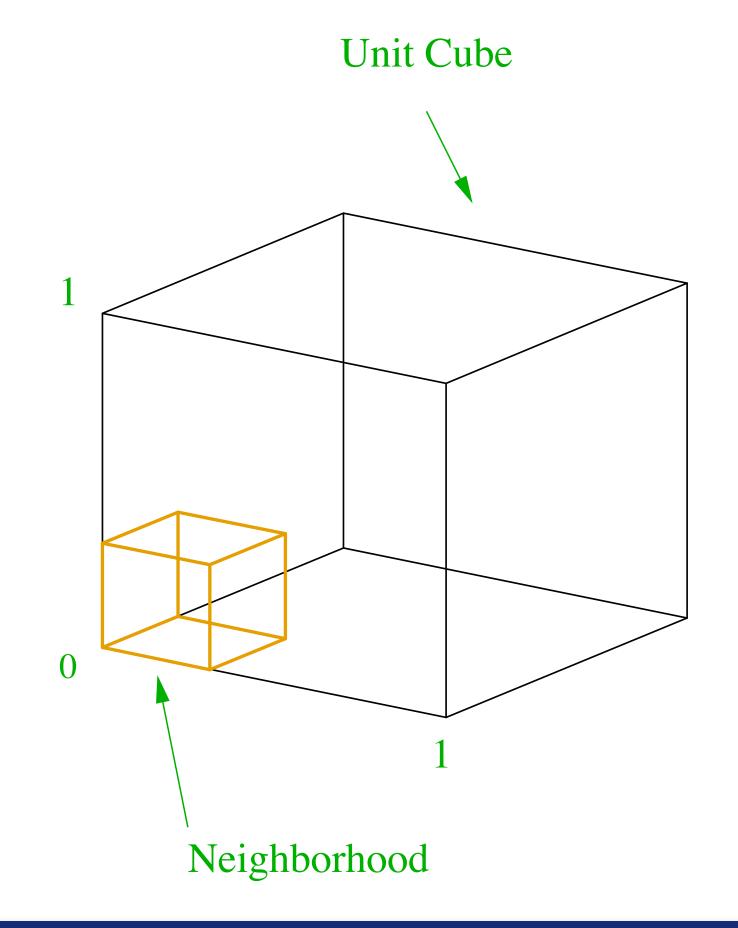
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$$\begin{array}{c} \alpha = 1 \% \\ d = 5 \end{array} \Rightarrow r \approx 0.4 \qquad \begin{array}{c} \alpha = 1 \% \\ d = 10 \end{array} \Rightarrow r \approx 0.63 \end{array}$$

Imagine all points lie in d-dimensional unit cube $[0,1]^d$

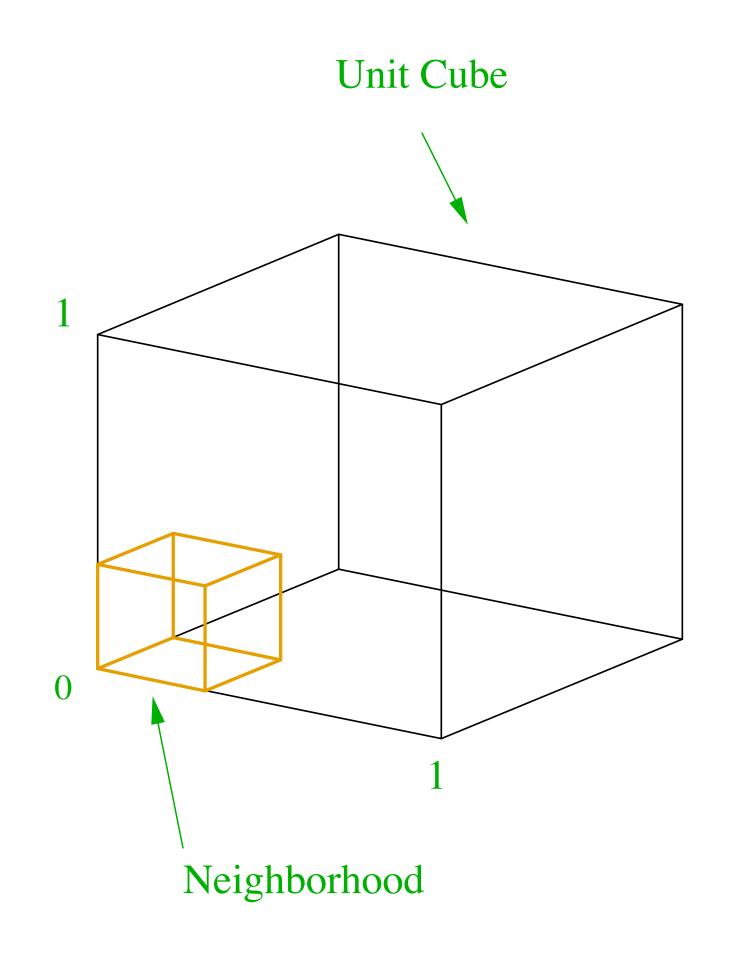


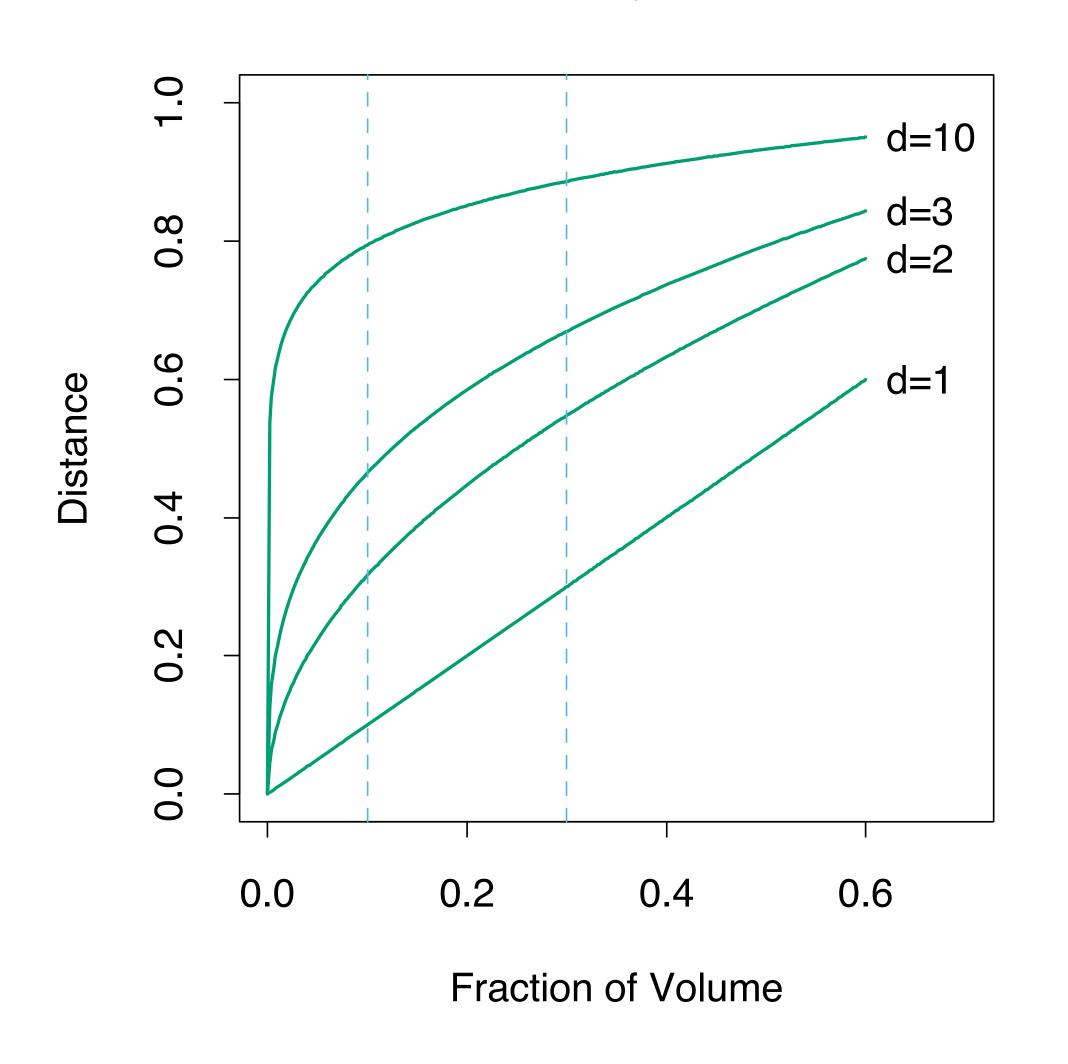
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What fraction of the total volume does this cube cover?

$$\begin{array}{c} \alpha = 10 \% \\ d = 10 \end{array} \Rightarrow r \approx 0.8$$





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How to choose r such that this probability is 1/2?

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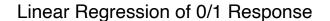
$$r = \sqrt[d]{1 - \sqrt[s]{\frac{1}{2}}}$$

Example:

$$d = 10$$

$$\Rightarrow r \approx 0.52$$

$$s = 500$$



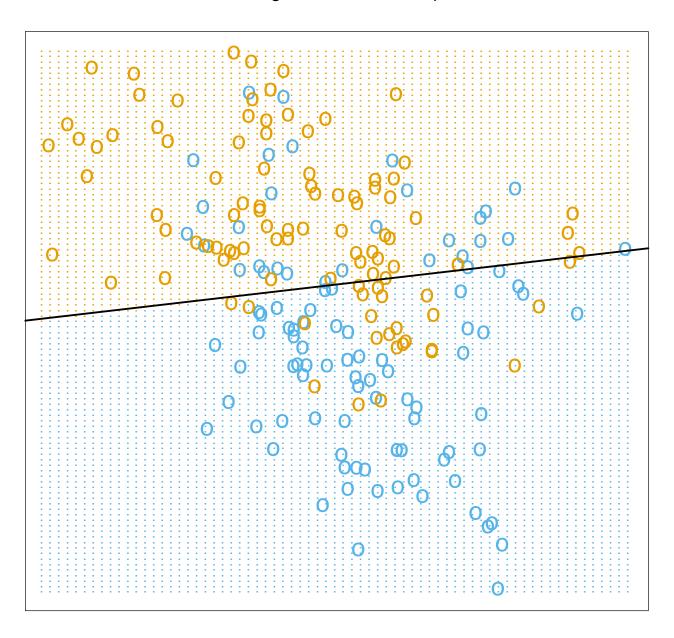


FIGURE 2.1. A classification example in two dimensions. The classes are coded as a binary variable (BLUE = 0, ORANGE = 1), and then fit by linear regression. The line is the decision boundary defined by $x^T \hat{\beta} = 0.5$. The orange shaded region denotes that part of input space classified as ORANGE, while the blue region is classified as BLUE.



Suppose $\{(x_i, y_i)\}_{i=1}^s$ are points sampled from the distribution \mathscr{D}



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$$p(y=c\,|\,x,\mathcal{D},K) = \frac{1}{K}\sum_{i\in N_K(x,\mathcal{D})}\iota(y_i=c)$$
 with
$$\iota(z) := \begin{cases} 1 & \text{if } z \text{ is true} \\ 0 & \text{if } z \text{ is false} \end{cases}$$



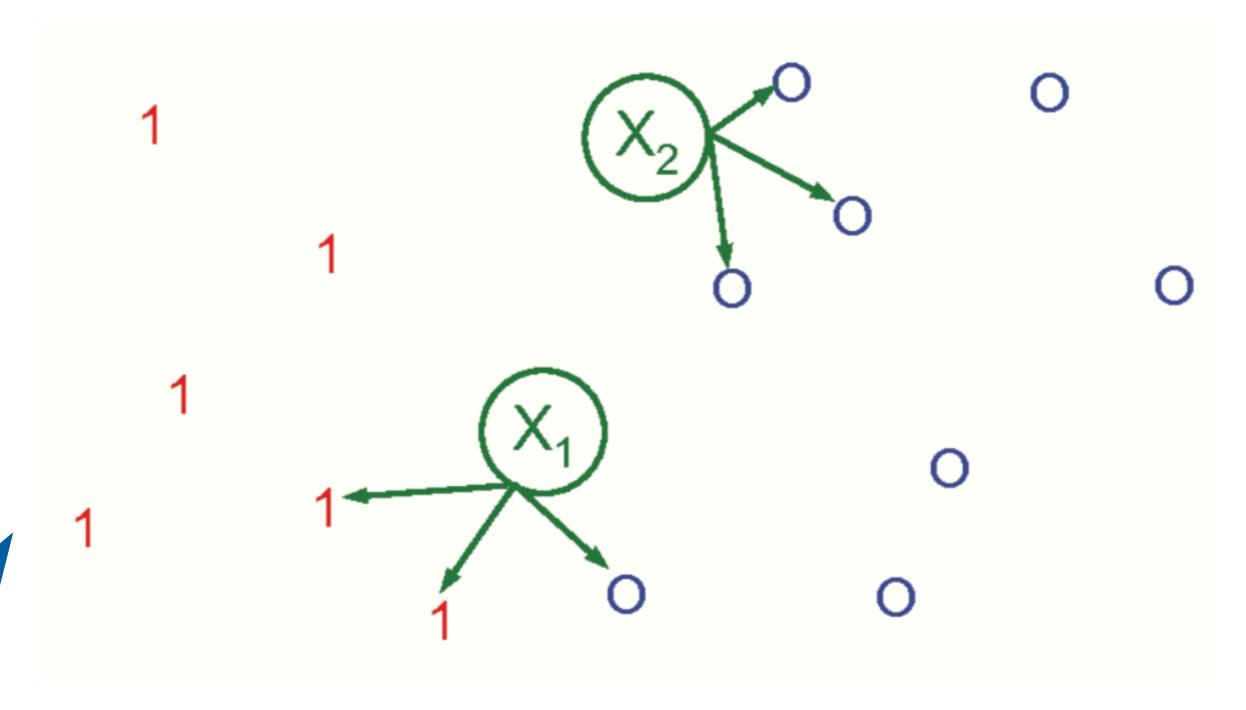
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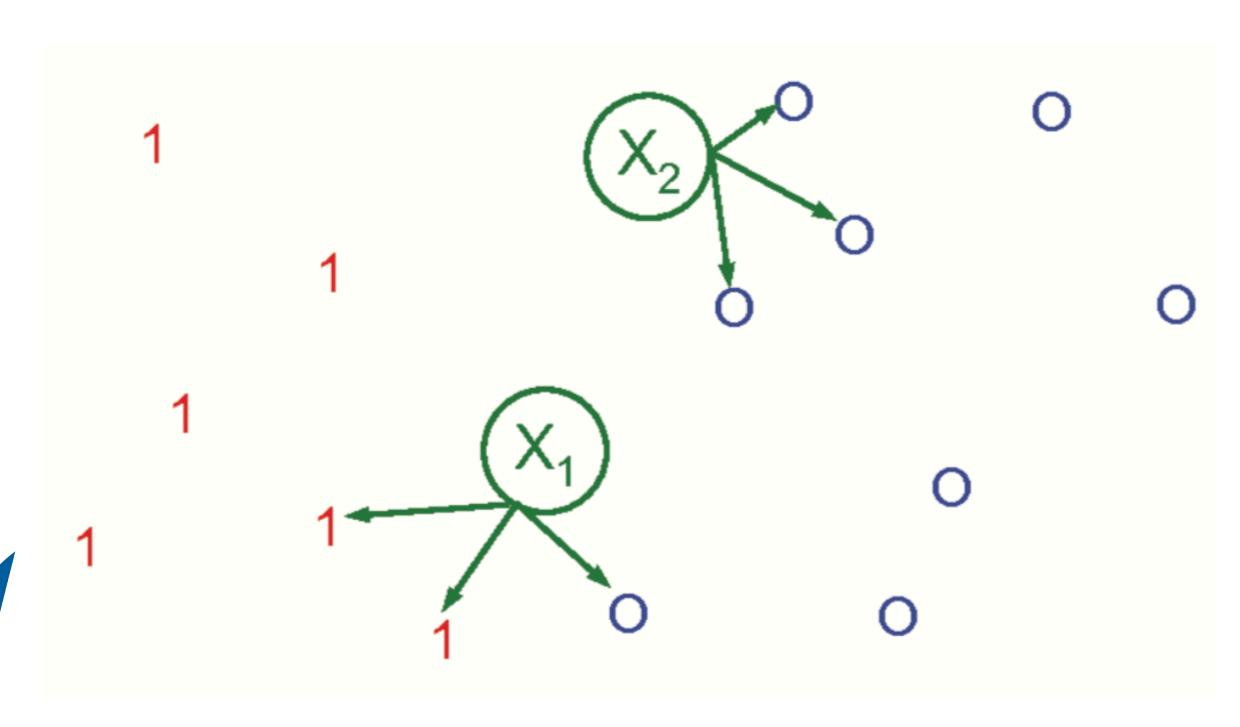
Subsequently we set $f(x) = \arg \max_{c} p(y = c \mid x, \mathcal{D}, K)$

Example:



Kevin Murphy. Machine Learning

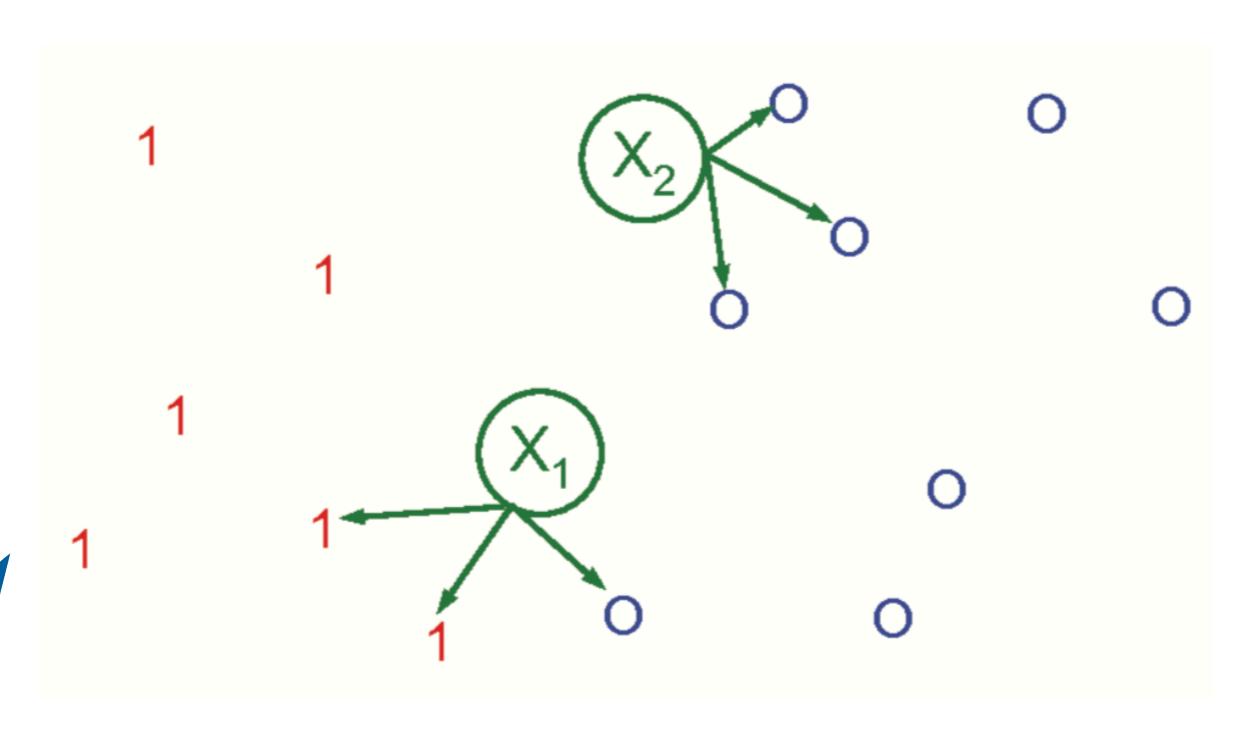
Example:



$$p(y = 1 | x_1, \mathcal{D}, K = 3) = 2/3$$

$$p(y = 0 | x_1, \mathcal{D}, K = 3) = 1/3$$

Example:



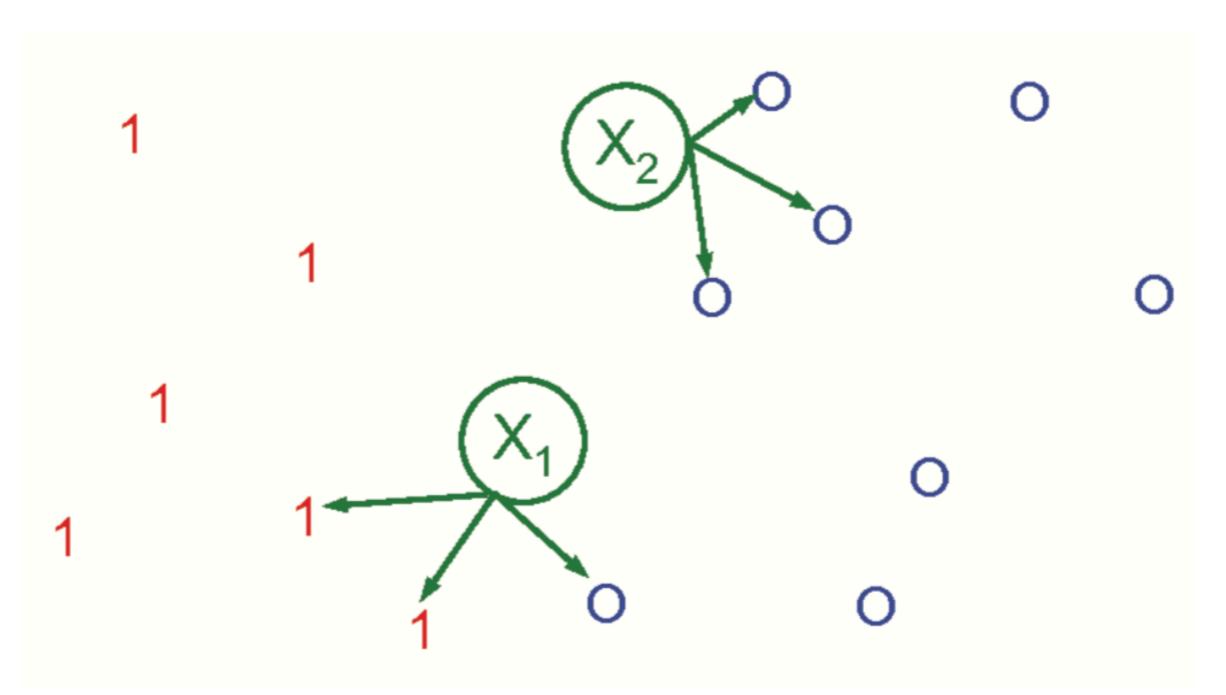
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$$\Rightarrow f(x_1) = 1$$



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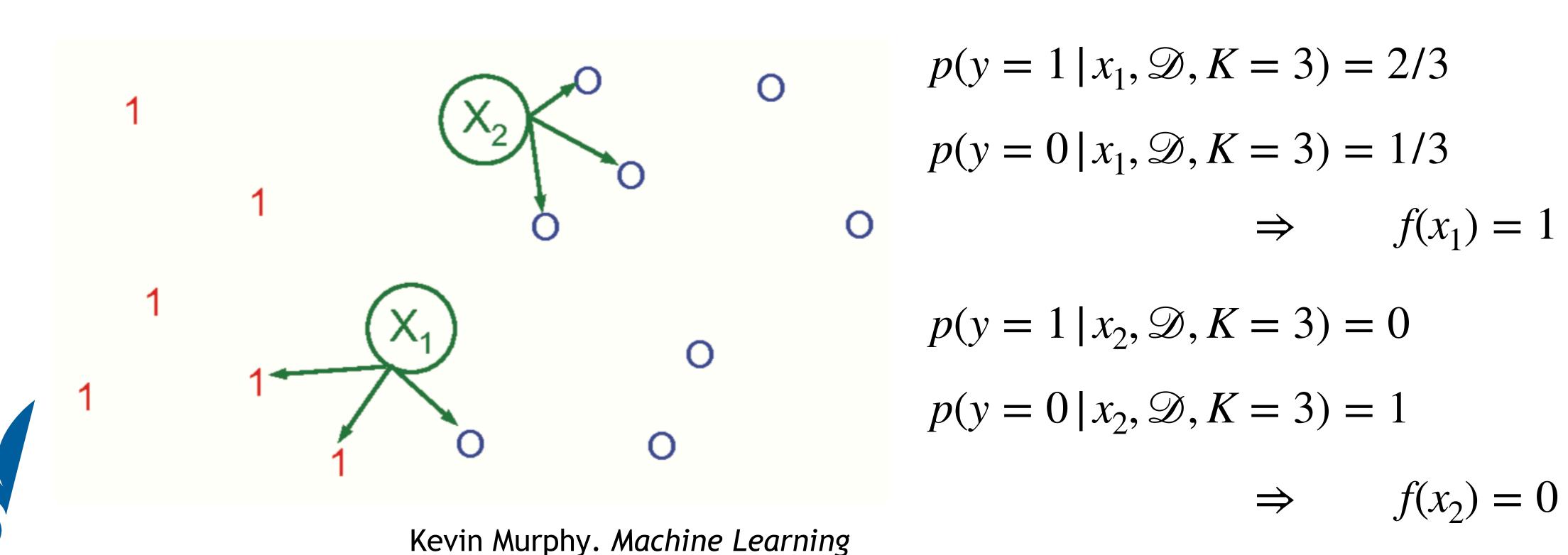
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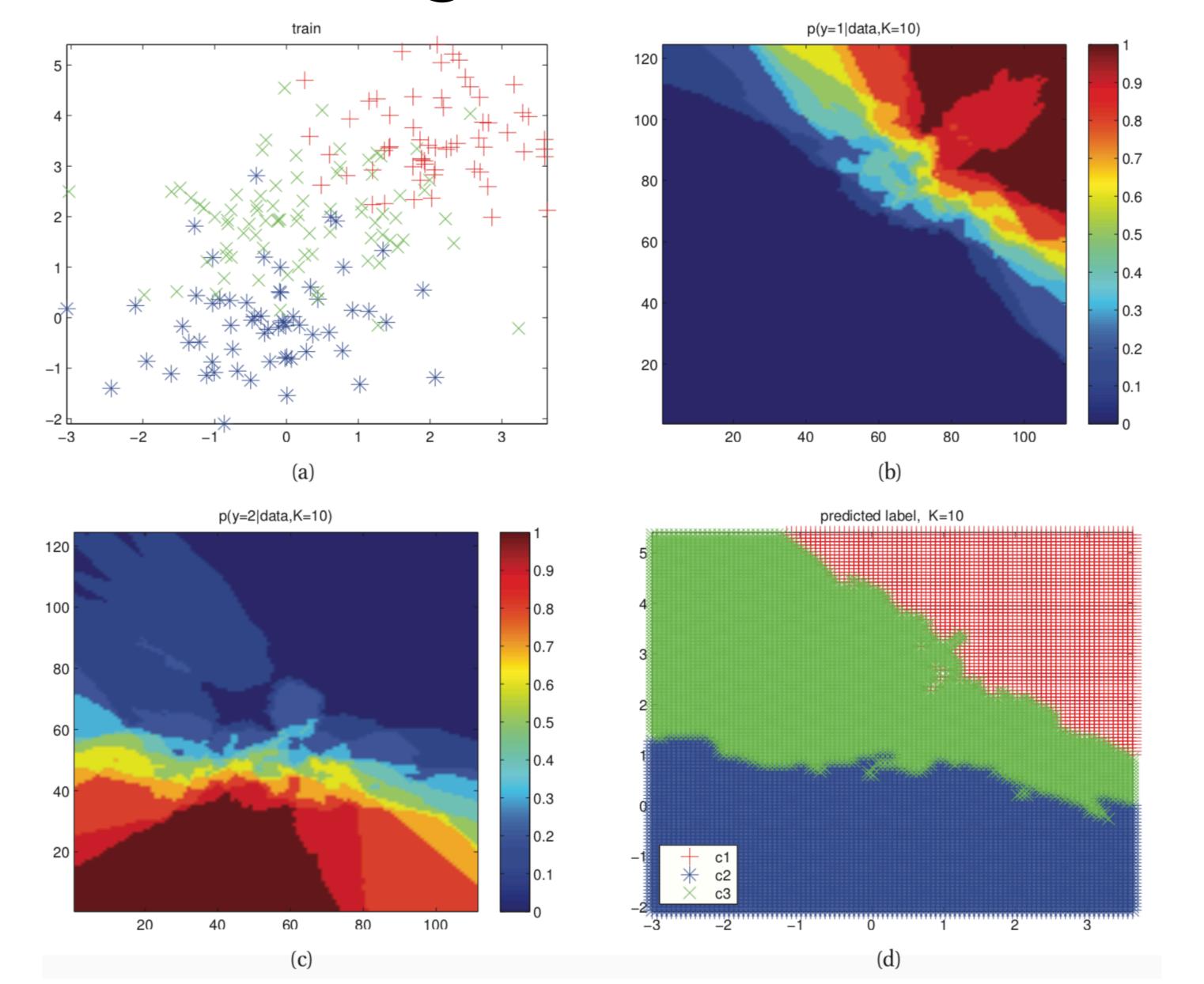
$$p(y = 1 | x_2, \mathcal{D}, K = 3) = 0$$

$$p(y = 0 | x_2, \mathcal{D}, K = 3) = 1$$



Example:







1-Nearest Neighbor Classifier

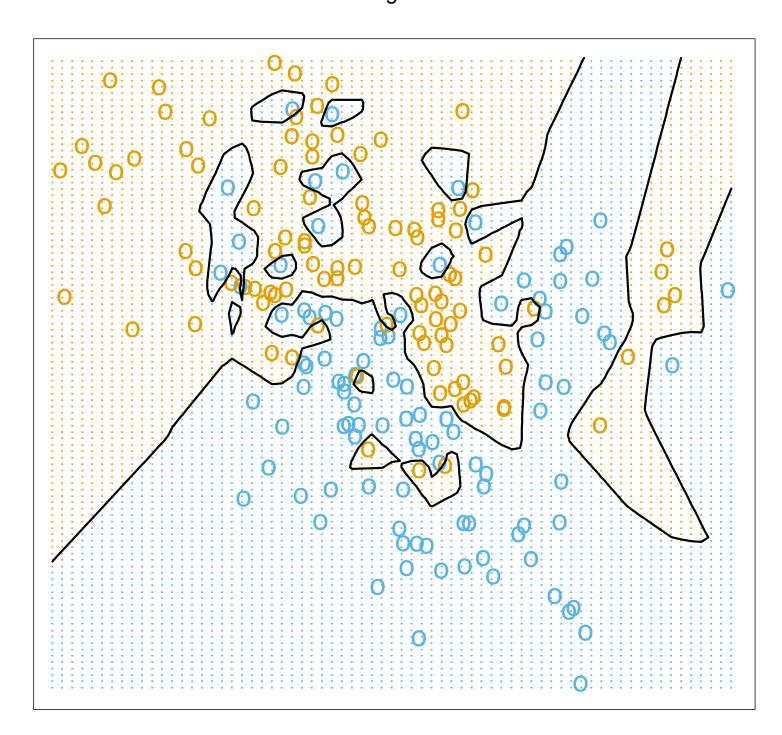


FIGURE 2.3. The same classification example in two dimensions as in Figure 2.1. The classes are coded as a binary variable (BLUE = 0, ORANGE = 1), and then predicted by 1-nearest-neighbor classification.



15-Nearest Neighbor Classifier

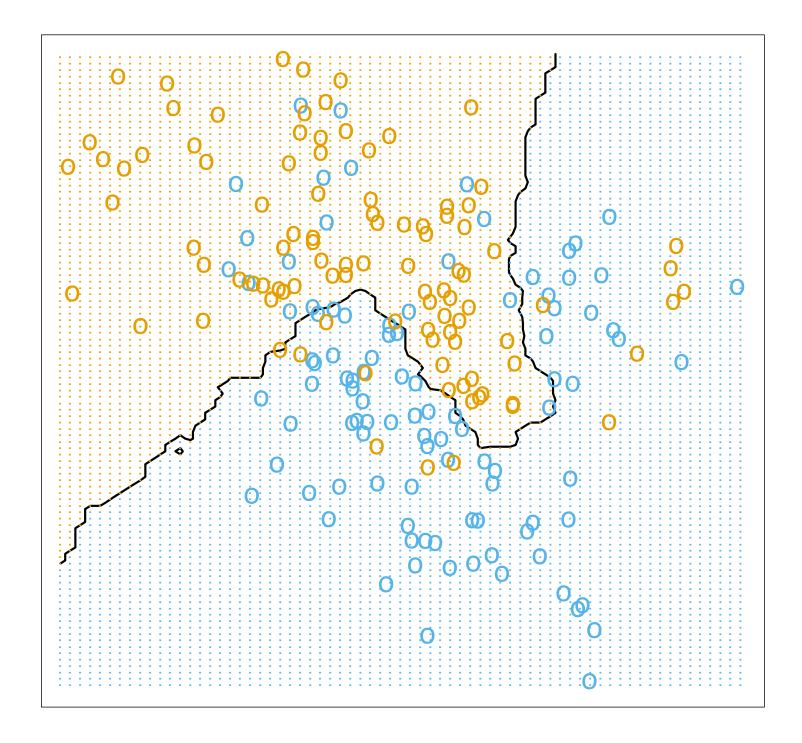


FIGURE 2.2. The same classification example in two dimensions as in Figure 2.1. The classes are coded as a binary variable (BLUE = 0, ORANGE = 1) and then fit by 15-nearest-neighbor averaging as in (2.8). The predicted class is hence chosen by majority vote amongst the 15-nearest neighbors.



Solve
$$\hat{w} = \arg\min_{w} \left\{ \frac{1}{2s} \sum_{i=1}^{s} |f(x_i, w) - y_i|^2 + \frac{\alpha}{2} ||w||^2 \right\}$$
 for $y_i = \begin{cases} 0 & \text{for class label } C_1 \\ 1 & \text{for class label } C_2 \end{cases}$



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 for $y_i = \begin{cases} 0 & \text{for class label } C_1 \\ 1 & \text{for class label } C_2 \end{cases}$

Then it is natural to decide

$$f(x_i, \hat{w}) < \frac{1}{2}$$

The predicted output is in class with label C_1

$$f(x_i, \hat{w}) > \frac{1}{2}$$

 $f(x_i, \hat{w}) > \frac{1}{2}$ \Rightarrow The predicted output is in class with label C_2



Solve
$$\hat{w} = \arg\min_{w} \left\{ \frac{1}{2s} \sum_{i=1}^{s} |f(x_i, w) - y_i|^2 + \frac{\alpha}{2} ||w||^2 \right\}$$
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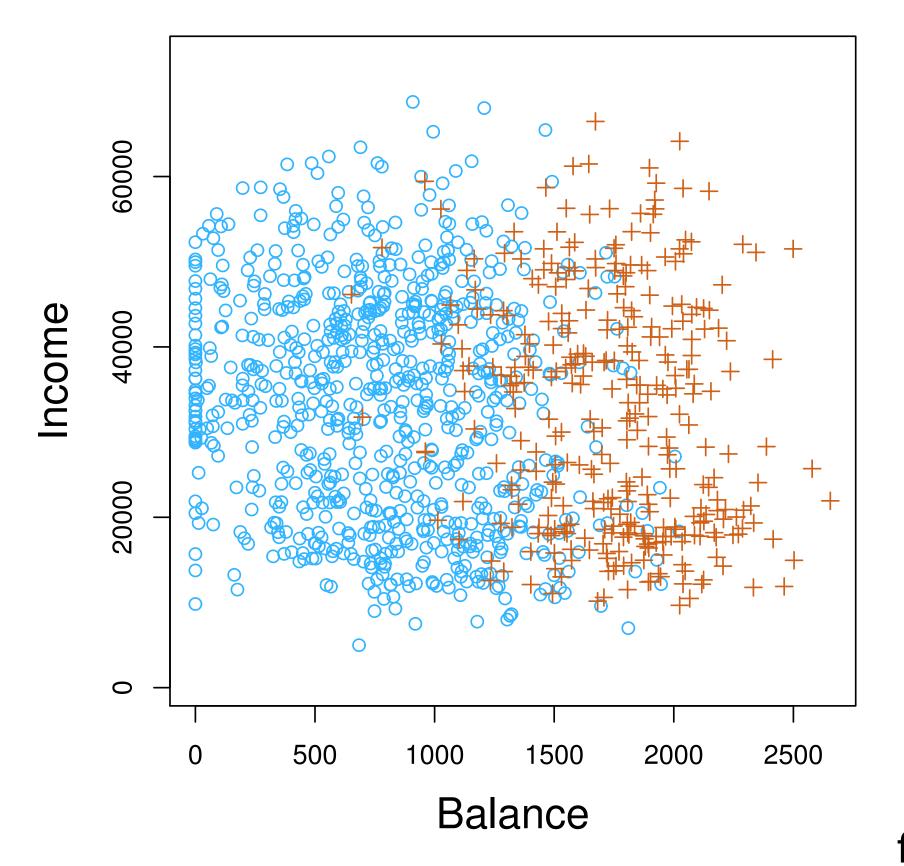
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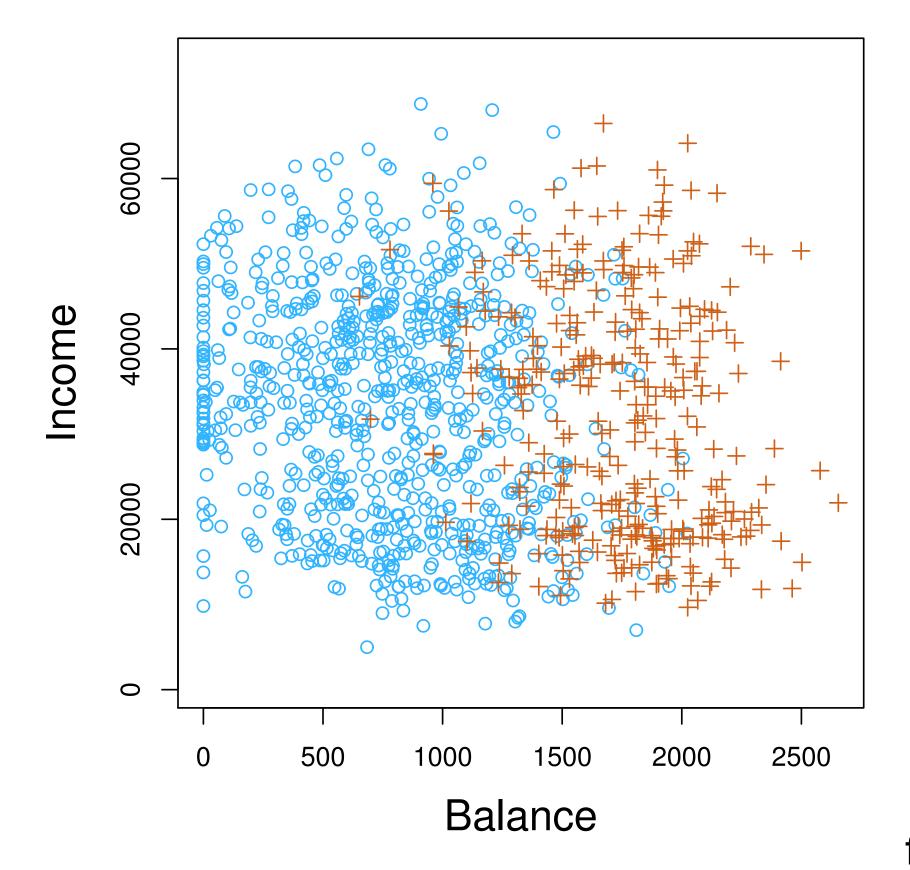


Example: credit default



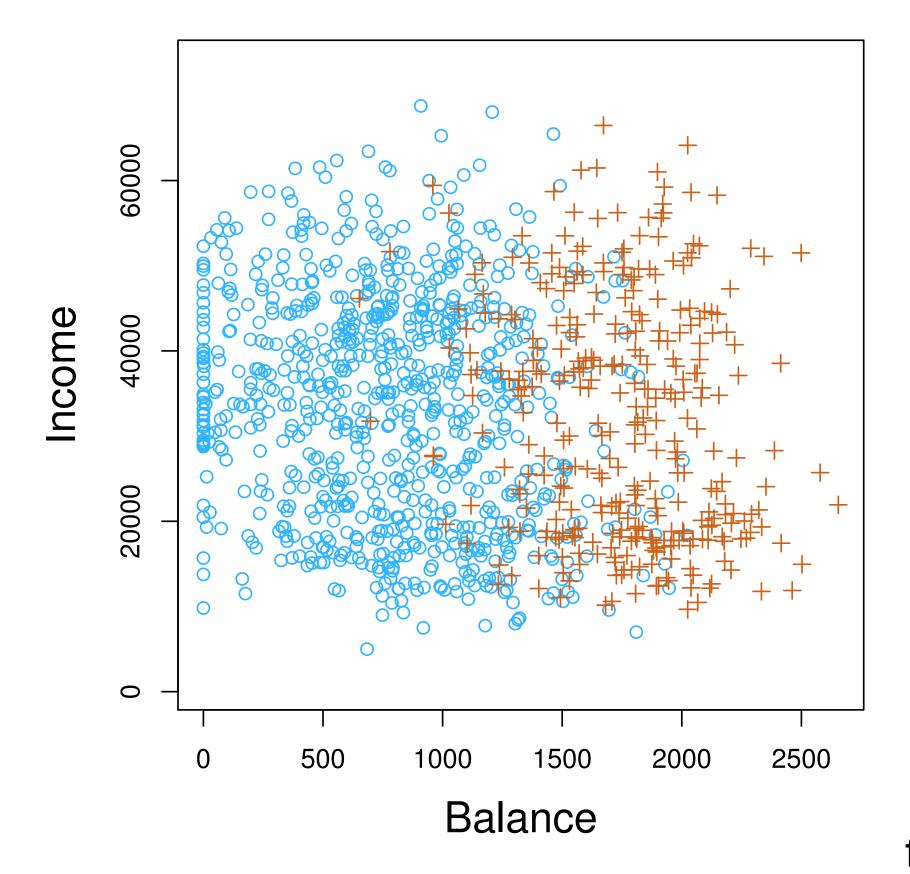
- + = individual who defaulted on their credit card payments
- o = individual who did not default on their credit card payments

Example: credit default



It seems that the balance is the dominant contributing factor towards predicting a default

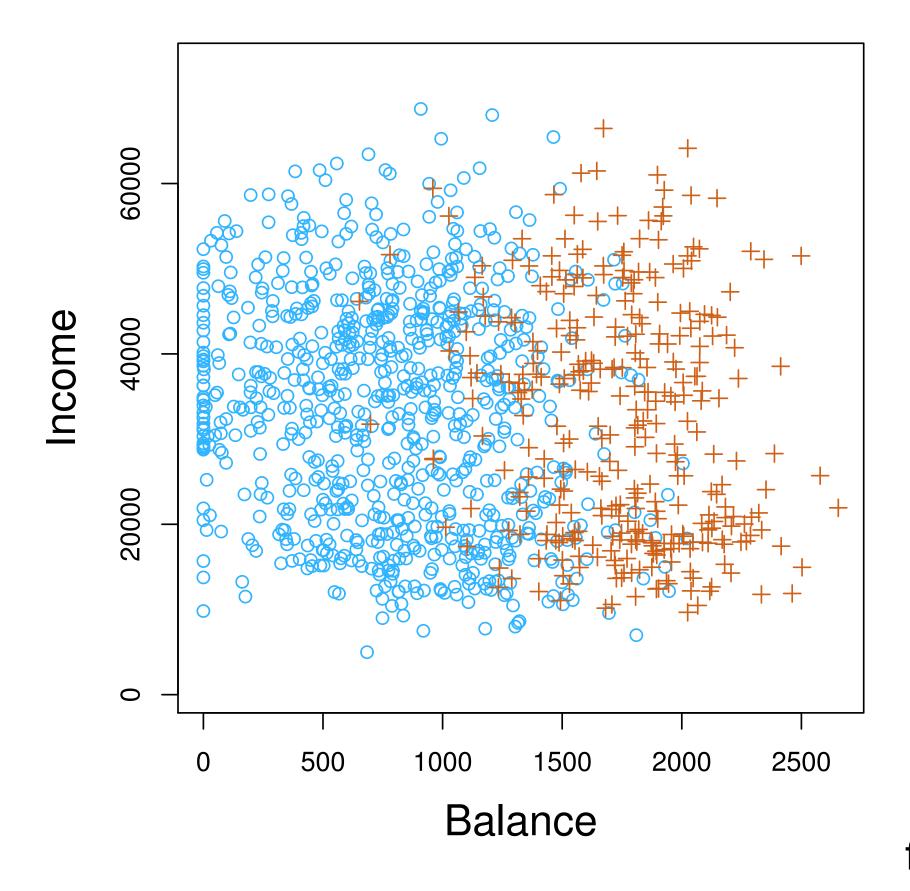
Example: credit default



It seems that the balance is the dominant contributing factor towards predicting a default

We therefore ignore the income for now and focus on the balance

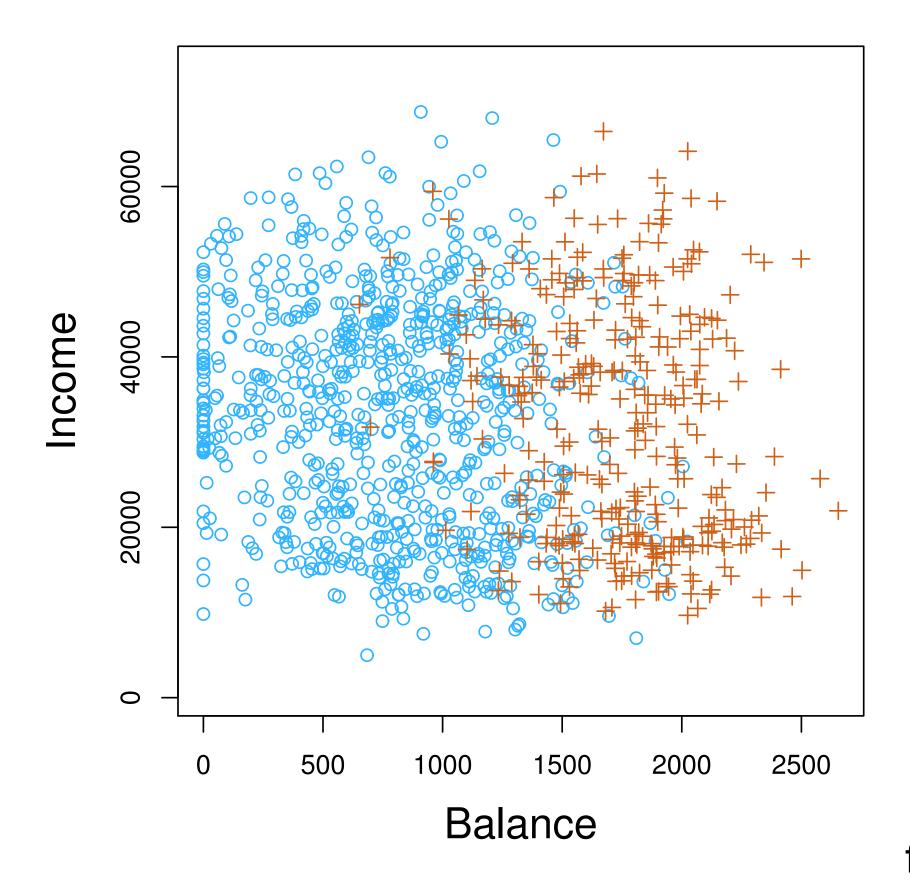
Example: credit default



Model assumption

$$f(x, w_0, w_1) = w_0 + w_1 x$$

Example: credit default



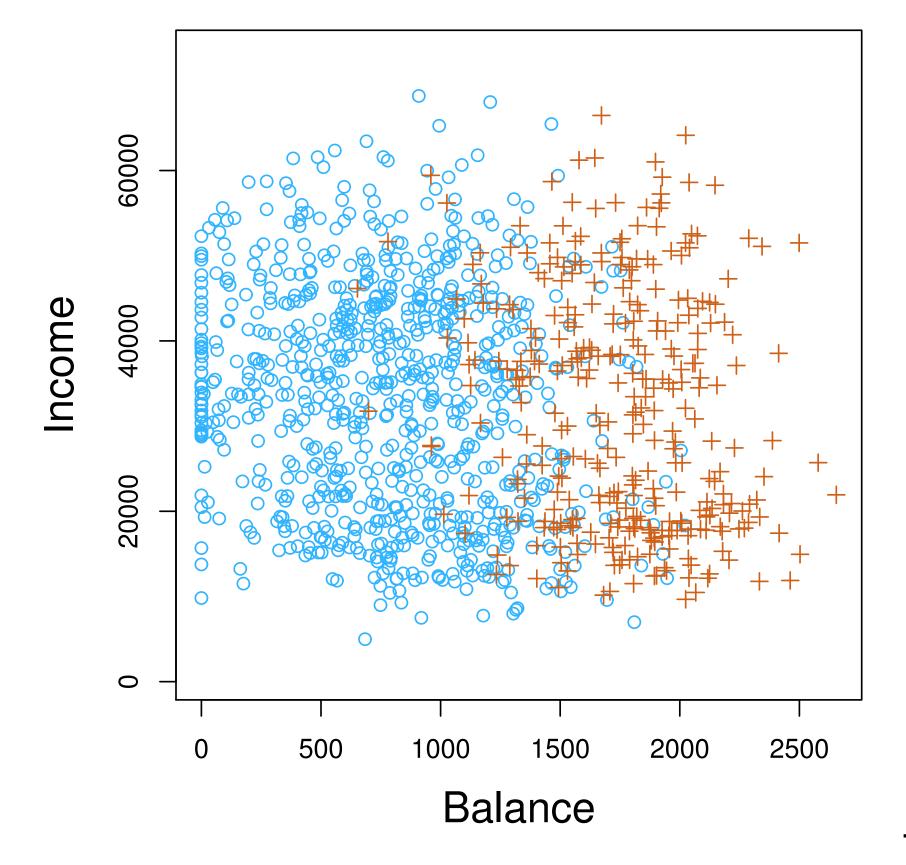
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Input $\{x_i\}_{i=1}^s$

 x_i = Balance of customer i

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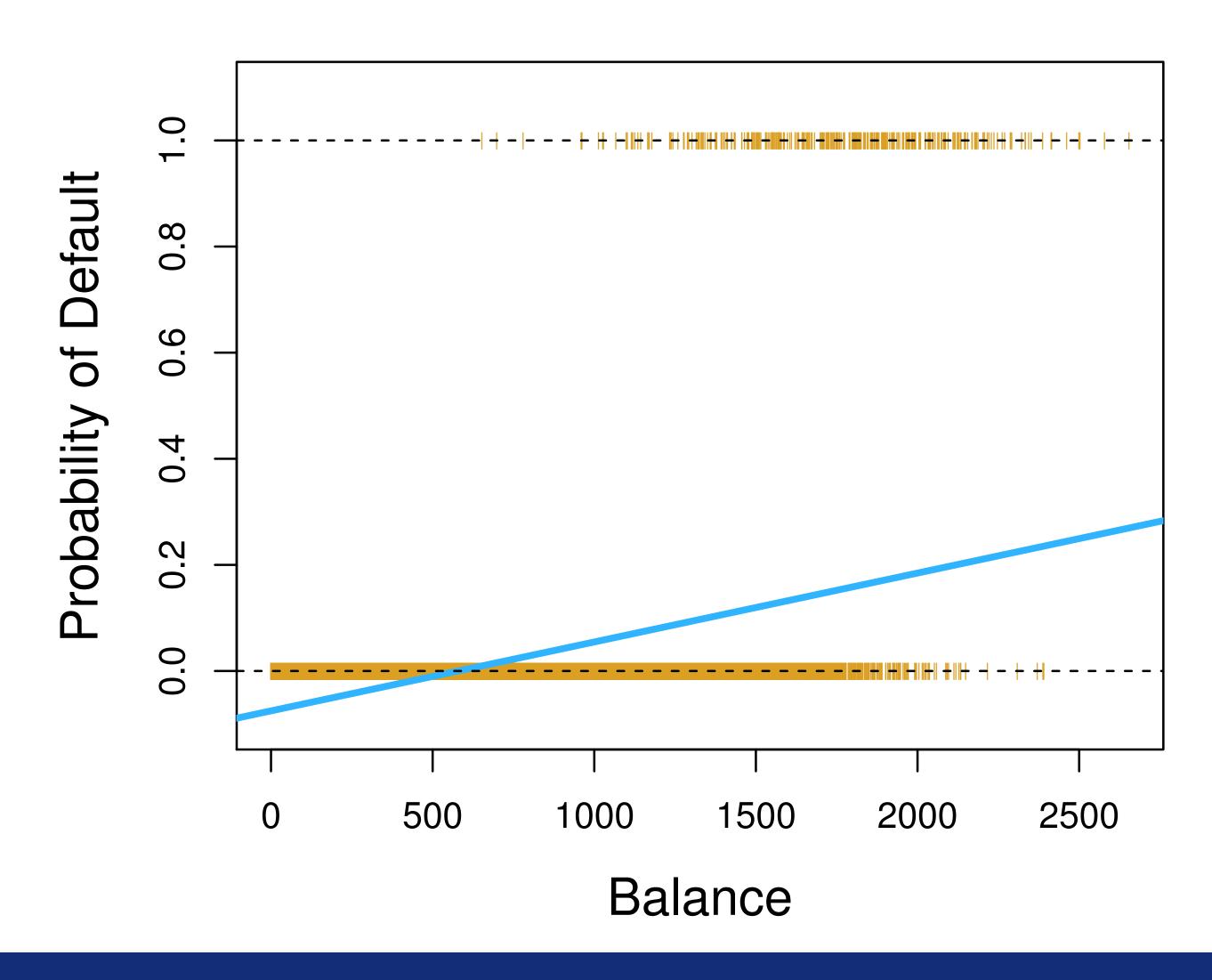
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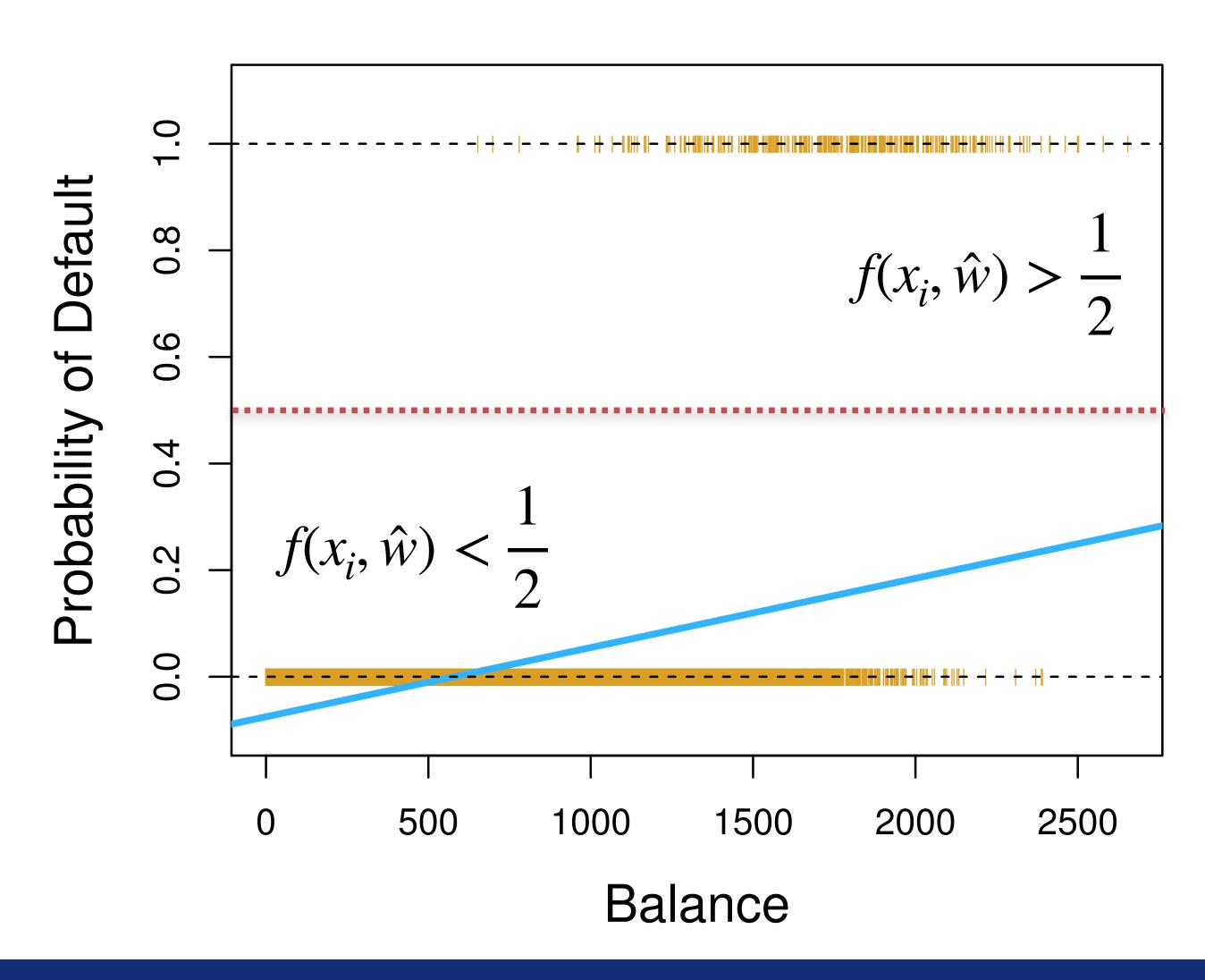
Output $\{y_i\}_{i=1}^s$

$$y_i = \begin{cases} 1 & \text{Customer } i \text{ did default} \\ 0 & \text{Customer } i \text{ did not default} \end{cases}$$

Regression outcome is as follows:

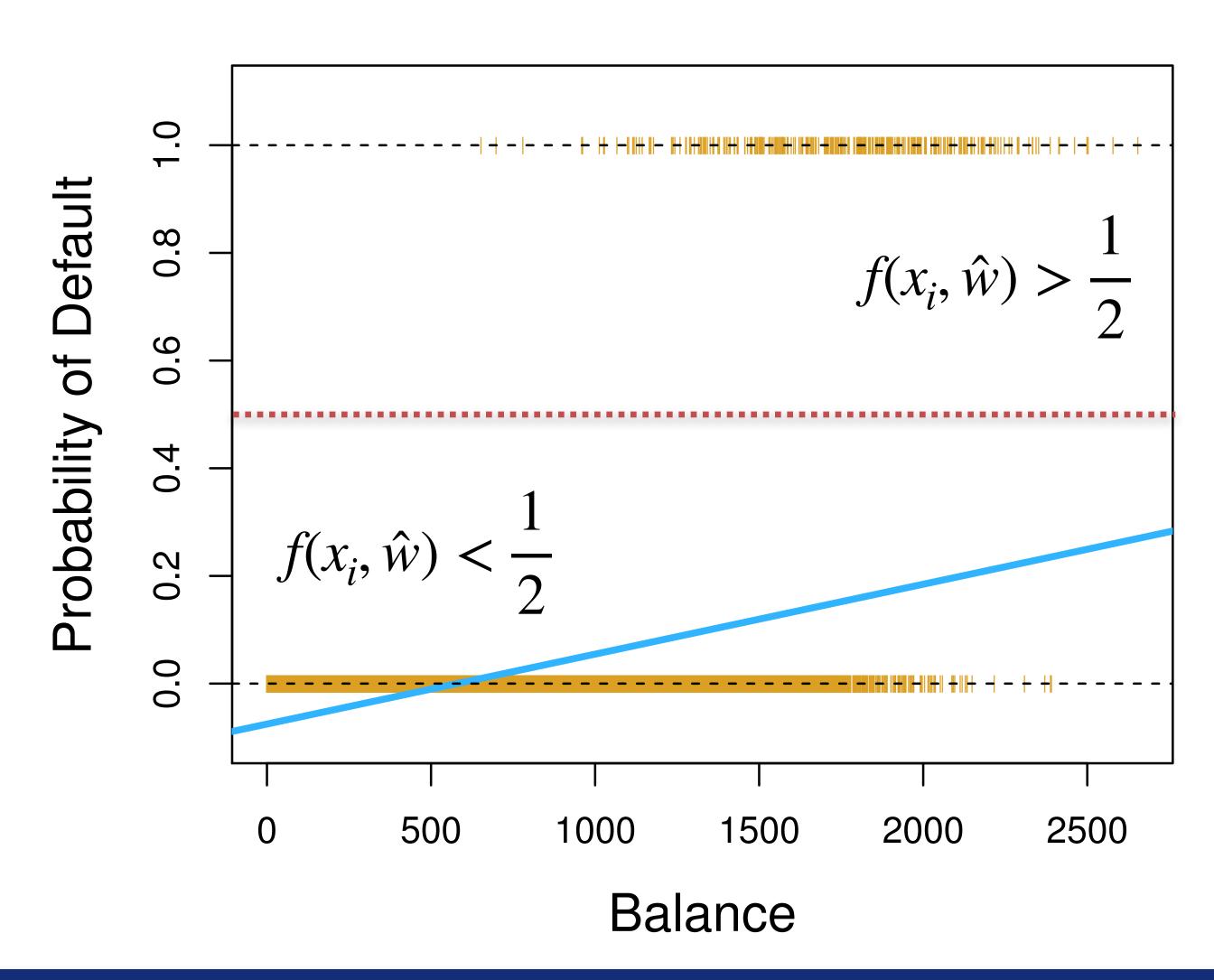


Regression outcome is as follows:



Regression outcome is as follows:

That didn't go according to plan. What went wrong?



'Position' of line will crucially depend



'Position' of line will crucially depend

- on how many points are in each class
- and where these points lie



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This is not a desirable property!



Why does this happen?

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MSE is not a good metric for these types of problems!