Queen Mary
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MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 4 (Sequences 2)

1. Which of the following sequences $(x_n)_{n=1}^{\infty}$ are monotonic? For those that are, state whether they are increasing or decreasing, and whether they are *strictly* increasing or decreasing. Give a brief justification in each case.

   (a) $x_n = \frac{n}{(n+1)(n+2)}$;

   (b) $x_n = \frac{n+1}{(n+2)(n+3)}$;

   (c) $x_n = \cos \pi n$;

   (d) $x_n = \lceil n^{1/2} \rceil$;

   Solution.

   (a) The first few terms are $x_1 = 1/6$, $x_2 = 1/6$, $x_3 = 3/20$ and $x_4 = 2/15$ so it looks like the sequence is decreasing, albeit slowly. We check this for general $n \in \mathbb{N}$:

   $$x_n - x_{n+1} = \frac{n}{(n+1)(n+2)} - \frac{n+1}{(n+2)(n+3)} = \frac{n-1}{(n+1)(n+2)(n+3)}.$$

   So we have $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$. In other words the sequence is indeed decreasing, but not strictly decreasing, since $x_1 = x_2$.

   (b) This is just the sequence from part (a), but started at x_2. It is thus strictly decreasing.

   (c) The first elements are $x_1 = -1$; $x_2 = +1$; $x_3 = -1$; $x_4 = +1$. In fact $x_n = -1$ when n is odd, and $x_n = +1$ when n is even. So, this sequence is neither increasing nor decreasing.

   (d) The sequence given by $x_n = \lceil n^{1/2} \rceil$ is an increasing sequence. To prove this, observe that we know that $\sqrt{n+1} > \sqrt{n}$ for all n, and from the definition of $\lceil y \rceil$ we know that if $y > z$ then $\lceil y \rceil \geq \lceil z \rceil$.

   However the sequence is *not* strictly increasing since, for example, $x_2 = \lceil 2^{1/2} \rceil = \lceil 1.414... \rceil = 2$ and $x_3 = \lceil 3^{1/2} \rceil = \lceil 1.732... \rceil = 2$.

2. For each of the following three sequences state whether they converge to a limit. If a sequence converges, state and prove the limit. (You may use any results from the lecture notes but you should state which result you are using.) If the sequence does not converge, find two subsequences that converge to different limits. Again, justify your answer by reference to results from the lecture notes.

(a) $(x_n)_{n=1}^{\infty}$, where $x_n = \big(1+\sin(n\pi/5)\big)^2\Big(\dfrac{n+1}{2n^2}\Big)$,

(b) $(y_n)_{n=1}^{\infty}$, where $y_n = \dfrac{2n^2+5(-1)^n}{4n^2+1}$,

(c) $(z_n)_{n=1}^{\infty}$, where $z_n = (-1)^n \left(\dfrac{n+3}{n+2}\right)$.

Solution. It might be useful in the following to remember that $(\frac{1}{n})$ converges to zero, that $(-1)^n$ is a sequence which does *not* converge and that $|\cos(x)| \leq 1$ for all $x \in \mathbb{R}$. Moreover, the following results from the lecture course are useful:

- Lemma 2.3: if $x_n \to 0$ and $|y_n| \leq |x_n|$ for all $n \in \mathbb{N}$, then $y_n \to 0$ ("dominated convergence").
- Theorem 2.19 (i) (aka Lemma 2.5 in the case $x = 0$): if $x_n \to x$ then $cx_n \to cx$.
- Theorem 2.19 (ii) (aka Theorem 2.8 in the case $x = 0 = y$): if $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$.
- Theorem 2.19 (iii) (aka Theorem 2.13 in the case $x = 0 = y$): if $x_n \to x$ and $y_n \to y$ then $x_n y_n \to xy$.
- Theorem 2.19 (iv): if $x_n \to x$ and $y_n \to y$ then $x_n/y_n \to x/y$ provided $y \neq 0$ and $y_n \neq 0$.
- Theorem 2.17: if $x_n \to 0$ and $|y_n| \leq M$ for all n then $x_n y_n \to 0$.

We can now solve the exercises.

(a) $(x_n)_{n=1}^{\infty}$, where
$$x_n = \big(1+\sin(n\pi/5)\big)^2\Big(\frac{n+1}{2n^2}\Big),$$
converges to 0.

Observe that x_n is the product of two factors. From the chain of inequalities
$$\big|(1+\sin(n\pi/5))^2\big| = |1+\sin(n\pi/5)|^2 \leq \big(1+|\sin(n\pi/5)|\big)^2 \leq 2^2 = 4$$
we see that the first factor is bounded. We can bound the second factor as follows:
$$\Big|\frac{n+1}{2n^2}\Big| = \frac{|n+1|}{|2n^2|} \leq \frac{n+n}{2n^2} = \Big|\frac{1}{n}\Big|.$$
We know that $(1/n)$ converges to 0, so the second factor converges to 0 by dominated convergence. Therefore (x_n) converges to zero by Theorem 2.17 (see above).

(b) $(y_n)_{n=1}^{\infty}$, where
$$y_n = \frac{2n^2+5(-1)^n}{4n^2+1},$$
converges to $\frac{1}{2}$.

First, divide through by n^2:

$$y_n = \frac{2 + 5(-1)^n n^{-2}}{4 + n^{-2}}.$$

We know that (n^{-2}) converges to 0 by comparison with (n^{-1}) (dominated convergence), and hence $(5n^{-2})$ converges to 0, by Lemma 2.5/Theorem 2.19(i). Since $|5(-1)^n n^{-2}| \leq 5n^{-2}$, we see that $(5(-1)^n n^{-2})$ converges to 0 (dominated convergence).

By Theorem 2.19(ii), the numerator converges to 2 and the denominator to 4. Finally, by Theorem 2.19(iv), the quotient converges to $\frac{1}{2}$.

(c) $(z_n)_{n=1}^{\infty}$, where

$$z_n = (-1)^n \left(\frac{n+3}{n+2}\right),$$

does not converge, but has subsequences converging to $+1$ and -1.

Let $\hat{z}_k = z_{2k}$ for all $k \in \mathbb{N}$. So $(\hat{z}_k)_{k=1}^{\infty} = (z_2, z_4, z_6, \ldots)$ is a subsequence of (z_n) consisting of the even terms. Note that

$$\hat{z}_k = \frac{2k+3}{2k+2} = 1 + \frac{1}{2k+2}.$$

and that $\frac{1}{2k+2} \leq \frac{1}{k}$. We saw that $\left(\frac{1}{k}\right)$ converges to 0, so $(\hat{z}_k)$ converges to 1 by dominated convergence and Theorem 2.19(ii).

Let $\tilde{z}_k = z_{2k-1}$ for all $k \in \mathbb{N}$. So $(\tilde{z}_k)_{k=1}^{\infty} = (z_1, z_3, z_5, \ldots)$ is a subsequence of (z_n) consisting of the odd terms. Note that

$$\tilde{z}_k = -\frac{2k+2}{2k+1} = -1 - \frac{1}{2k+1},$$

and that $\frac{1}{2k+1} \leq \frac{1}{k}$. So $(\hat{z}_k)$ converges to -1 by dominated convergence and Theorem 2.19(ii).

3. Prove the following “Sandwich principle” mentioned in the notes:

If (y_n) is *some sequence* and (x_n) and (z_n) are two other sequences with $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, then (y_n) converges as well and $\lim_{n\to\infty} y_n = L$.

Solution. We claim: If (y_n) is *some sequence* and (x_n) and (z_n) are two other sequences with $x_n \leq y_n \leq z_n$ and with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, then (y_n) converges as well and $\lim_{n\to\infty} y_n = L$.

Proof. Given $\varepsilon > 0$, the convergence of (x_n) and (z_n) to L implies that we can find some $N_x \in \mathbb{N}$ and $N_z \in \mathbb{N}$, such that for all $n > N_x$ we have $|x_n - L| < \varepsilon$ and for all $n > N_z$

we have $|z_n - L| < \varepsilon$. Note, in particular, that $x_n > L - \varepsilon$ and $z_n < L + \varepsilon$. Now set $N = \max\{N_x, N_z\}$. Then for $n > N$, we conclude that

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon.$$

It follows that $|y_n - L| < \varepsilon$ for all $n > N$, and thus $y_n \to L$ as $n \to \infty$. □

4. Let the sequence $(x_n)_{n=1}^{\infty}$ be defined inductively by

$$x_1 = 2 \qquad \text{and} \qquad x_{n+1} = \frac{x_n}{2} + \frac{1}{4 - x_n}. \tag{$\star$}$$

(a) Compute x_2 and x_3 (and maybe use a calculator to obtain approximations to the next few terms).

(b) Prove that $(x_n)_{n=1}^{\infty}$ is strictly decreasing. *Hint: the right hand side of ($\star$) is monotonically increasing in the range* $(-\infty, 4)$.

(c) Prove that 0 is a lower bound for $(x_n)_{n=1}^{\infty}$.

(d) Deduce that $(x_n)_{n=1}^{\infty}$ converges and compute the limit. (This goes a little beyond where we are in the module, but do the best you can.)

Solution.

(a) $x_2 = 1 + 1/2 = 3/2$ and $x_3 = 3/4 + 2/5 = 23/20$.

(b) Let $f(x) = x/2 + 1/(4 - x)$, so that $x_{n+1} = f(x_n)$. Observe that $f(x)$ is a monotonically increasing function in the range $(-\infty, 4)$. (In fact, it is monotonically increasing everywhere except at $x = 4$, where $f(x)$ is undefined.) We prove $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$ by induction on n. We saw in (a) that this inequality holds for $n = 1$. Now suppose $x_n - x_{n+1} \geq 0$ for some $n \in \mathbb{N}$. Then $x_{n+1} - x_{n+2} = f(x_n) - f(x_{n+1}) \geq 0$ since $x_n \geq x_{n+1}$ and f is monotonically increasing.

(c) Again, use induction on n. The claim is true for $n = 1$. It is clear that $f(x) \geq 0$ for all $x \in [0, 4)$. We know from part (b) that $x_n \leq 2 < 4$ for all $n \in \mathbb{N}$. Suppose that $x_n \geq 0$ for some $n \in \mathbb{N}$. Then $x_{n+1} = f(x_n) \geq 0$. This deals with the inductive step.

(d) By Theorem 2.26, $(x_n)_{n=1}^{\infty}$ converges to a real number, say x. Consider the identity $x_{n+1} = f(x_n)$. Taking the limit of both sides, we obtain $x = f(x)$. (We are using the fact here that f is continuous in the range $(-\infty, 4)$, a concept we don't officially encounter until later in the module.) Substituting the explicit form for f yields the quadratic $x^2 - 4x + 2 = 0$, which has solutions $x = 2 \pm \sqrt{2}$. So $(x_n)_{n=1}^{\infty}$ must converge to $2 - \sqrt{2}$, since the other root is greater than 2.

As an aside, the next few terms of the series are $x_4 = 0.925877$, $x_5 = 0.788235$, $x_6 = 0.705473$ and $x_7 = 0.656270$, so convergence to $2 - \sqrt{2} \approx 0.585786$ does not seem particularly rapid.

5. For each of the following sequences, identify all its accumulation points. For each accumulation point $a \in \mathbb{R}$ give a subsequence that converges to a.

   (a) $(x_n)_{n=1}^{\infty}$, where $x_n = n^{-1}\cos(n\pi/2)$,

   (b) $(y_n)_{n=1}^{\infty}$, where $y_n = \cos(n\pi/2)$, and

   (c) $(z_n)_{n=1}^{\infty}$, where $z_n = n\cos(n\pi/2)$.

   Solution.

   (a) The only accumulation point is 0, since the sequence itself converges to 0 (e.g., by comparison with (n^{-1}) using dominated convergence). The trivial subsequence $(x_n)_{n=1}^{\infty}$ converges to 0 as does $(x_1, x_3, x_5, \ldots) = (0, 0, 0, \ldots)$.

   (b) (y_n) is the sequence $(0, -1, 0, +1, 0, -1, 0, +1, \ldots)$, so the accumulation points are $= 1$, 0 and 1. We can see this by considering the three subsequences $(y_2, y_6, y_{10}, \ldots) = (-1, -1, -1, \ldots)$, $(y_1, y_3, y_5, \ldots) = (0, 0, 0, \ldots)$ and $(y_4, y_8, y_{12}, \ldots) = (1, 1, 1, \ldots)$. There are no other accumulation points. (For every $a \in \mathbb{R} \setminus \{-1, 0, 1\}$ there exists $\varepsilon > 0$ such that $|y_n - a| \geq \varepsilon$ for all $n \in \mathbb{N}$.

   (c) The only accumulation point is 0, corresponding to the subsequence $(z_1, z_3, z_5, \ldots) = (0, 0, 0, \ldots)$. No $a \in \mathbb{R} \setminus \mathbb{Z}$ can be an accumulation point by the same argument as used in (b). For any $a \in \mathbb{Z}\setminus\{0\}$ there is at most one term z_n of the sequence satisfying $|z_n - a| < \varepsilon = \frac{1}{2}$.

6. Consider the sequence $(x_n)_{n=1}^{\infty}$ defined by $x_n = (-1)^n(1 + 1/n)$. Define $b_k = \sup_{n \geq k} x_n = \sup\{x_n : n \in \mathbb{N} \text{ and } n \geq k\}$.

   (a) Evaluate the first six terms $b_1, b_2, \ldots, b_6$ of the sequence $(b_k)_{k=1}^{\infty}$, leaving the result as an exact rational number.

   (b) Give the general form for b_k, treating separately the cases k even and k odd.

   (c) Show that the sequence $(b_k)_{k=1}^{\infty}$ converges and state the limit of the sequence.

   (d) Verify that $(b_k)_{k=1}^{\infty}$ is decreasing.

   Solution.

   (a)

$$
\begin{aligned}
b_1 &= \sup\{-2, \tfrac{3}{2}, -\tfrac{4}{3}, \tfrac{5}{4}, -\tfrac{6}{5}, \tfrac{7}{6}, \ldots\} = \tfrac{3}{2}, \\
b_2 &= \sup\{\tfrac{3}{2}, -\tfrac{4}{3}, \tfrac{5}{4}, -\tfrac{6}{5}, \tfrac{7}{6}, \ldots\} = \tfrac{3}{2}, \\
b_3 &= \sup\{-\tfrac{4}{3}, \tfrac{5}{4}, -\tfrac{6}{5}, \tfrac{7}{6}, \ldots\} = \tfrac{5}{4}, \\
b_4 &= \sup\{\tfrac{5}{4}, -\tfrac{6}{5}, \tfrac{7}{6}, \ldots\} = \tfrac{5}{4}, \\
b_5 &= \sup\{-\tfrac{6}{5}, \tfrac{7}{6}, \ldots\} = \tfrac{7}{6}, \\
b_6 &= \sup\{-\tfrac{6}{5}, \tfrac{7}{6}, \ldots\} = \tfrac{7}{6}.
\end{aligned}
$$

(b) In general, $b_k = (k+2)/(k+1)$ when n is odd, and $b_k = (k+1)/k$ when n is even.

(c) When k is odd $|b_k - 1| = 1/(k+1) \leq 1/k$ and when k is even, $|b_k - 1| = 1/k$; either way, $|b_k - 1| \leq 1/k$. Since $(1/k)_{k=1}^{\infty}$ converges to 0, it follows that $(b_k)_{k=1}^{\infty}$ converges to 1.

(d) If k is odd then $b_k - b_{k+1} = (k+2)/(k+1) - (k+2)/(k+1) = 0$ and when k is even $b_k - b_{k+1} = (k+1)/k - (k+3)/(k+2) = 2/(k(k+2)) \geq 0$. So b_k is decreasing (but not strictly decreasing). Note that, as mentioned in lectures, the sequence (b_k) is necessarily decreasing, since the terms are defined as suprema over smaller and smaller sets.

7. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers. Recall what it means for $(x_n)_{n=1}^{\infty}$ to be a *Cauchy sequence*.

(a) Using only the definition, but not any results proved in the course, prove that $(x_n)_{n=1}^{\infty}$ given by
$$x_n = 2 + \frac{1}{3n^2}$$
is a Cauchy sequence.

(b) Using only the definition, but not any results proved in the course, prove that $(x_n)_{n=1}^{\infty}$ given by
$$x_n = \sum_{k=1}^{n} \frac{3}{k}$$
is *not* a Cauchy sequence.

Solution. This is actually a question from the May 2015 Exam. Recall that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence iff

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m > N\ :\ |x_n - x_m| < \varepsilon.$$

(a) *Proof.* Given $\varepsilon > 0$, pick $N = \lceil \frac{1}{\varepsilon} \rceil$. Then for all $m, n > N$, we obtain from the triangle inequality
$$|x_n - x_m| = \left| \frac{1}{3n^2} - \frac{1}{3m^2} \right| \leq \frac{1}{3n^2} + \frac{1}{3m^2} < \frac{2}{3N^2} \leq \frac{1}{N^2} \leq \frac{1}{N} \leq \varepsilon,$$
so $|x_n - x_m| < \varepsilon$. □

(b) We have to prove the negation of the above quantifier expression, i.e.
$$\exists \varepsilon > 0\ \forall N \in \mathbb{N}\ \exists n, m > N\ :\ |x_n - x_m| \geq \varepsilon.$$

Proof. Pick $\varepsilon = 1$. Given $N \in \mathbb{N}$, there is some $l \in \mathbb{N}$ such that $2^l > N$. We then pick $n = 2^{l+1}$ and $m = 2^l$ for this l. As

$$|x_n - x_m| = \sum_{k=m+1}^{n} \frac{3}{k}$$

is a sum consisting of 2^l elements which are all at least $\frac{3}{2^{l+1}}$, we can estimate this by

$$|x_n - x_m| \geq 2^l \cdot \frac{3}{2^{l+1}} = \frac{3}{2} \geq 1 = \varepsilon$$

which proves the claim. □

8. Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and let $(y_n)_{n=1}^{\infty}$ be the sequence defined by $y_n = x_{n+1}$ for each $n \in \mathbb{N}$. Prove, using only the definition of convergence:

   (a) If $(x_n)_{n=1}^{\infty}$ converges to x, then $(y_n)_{n=1}^{\infty}$ converges to x.

   (b) If $(y_n)_{n=1}^{\infty}$ converges to x, then $(x_n)_{n=1}^{\infty}$ converges to x.

Solution. Let $y_n = x_{n+1}$ for each $n \in \mathbb{N}$.

(a) If $(x_n)_{n=1}^{\infty}$ converges to x, then $(y_n)_{n=1}^{\infty}$ converges to x.

Proof. We must show that

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N \; : \; |y_n - x| < \varepsilon.$$

So given any $\varepsilon > 0$, we must find $N \in \mathbb{N}$ such that

$$\forall n > N \; : \; |x_{n+1} - x| < \varepsilon. \qquad (*)$$

So suppose we have been given $\varepsilon > 0$ (by the demon). Since $(x_n)_{n=1}^{\infty}$ converges to x, we know that there exists N_x such that $\forall n > N_x \; : \; |x_n - x| < \varepsilon$. But whenever $n > N_x$, it is also true that $n + 1 > N_x$. So we may take $N = N_x$ and then $(*)$ will be true. □

(b) If $(y_n)_{n=1}^{\infty}$ converges to x, then $(x_n)_{n=1}^{\infty}$ converges to x.

Proof. We must show that

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N \; : \; |x_n - x| < \varepsilon.$$

So given any $\varepsilon > 0$, we must find $N \in \mathbb{N}$ such that

$$\forall n > N \; : \; |x_n - x| < \varepsilon. \qquad (*)$$

So suppose we have been given $\varepsilon > 0$ (by the demon). Since $(y_n)_{n=1}^{\infty}$ converges to x, we know that there exists N_y such that $\forall n > N_y \; : \; |y_n - x| = |x_{n+1} - x| < \varepsilon$. But whenever $n > N_y$, we have $n + 1 > N_y + 1$. So we may take $N = N_y + 1$. Then for all $n > N$, we have $n - 1 > N_y$ and thus $|y_{n-1} - x| = |x_n - x| < \varepsilon$, i.e. $(*)$ will be true. □