Suppose that we have m failure times, $x = (x_1, \ldots, x_m)$, for a light bulb type A and n failure times, $y = (y_1, \ldots, y_n)$ for a light bulb type B. We assume that the samples x and y are independent of each other, $x = (x_1, \ldots, x_m)$ is an iid sample from Exponential(λ_1) and $y = (y_1, \ldots, y_n)$ is an iid sample from Exponential(λ_2), where $\lambda_1 > 0$ and $\lambda_2 > 0$ are unknown. Let gamma(a_1, b_1) and gamma(a_2, b_2) be independent priors on λ_1 and λ_2, respectively.

1. Find the joint posterior density, $p(\lambda_1, \lambda_2 \mid x, y)$ of λ_1 and λ_2.
2. How would you estimate $d = \frac{\lambda_1}{\lambda_2}$ by simulations?

Solution:

1. The joint posterior, $p(\lambda_1, \lambda_2 \mid x, y)$ of λ_1 and λ_2 is

$$\begin{aligned} p(\lambda_1, \lambda_2 \mid x, y) &\propto \text{prior} \times \text{likelihood} \\ &= p(\lambda_1, \lambda_2) \times p(x, y \mid \lambda_1, \lambda_2), \end{aligned}$$

where $p(\lambda_1, \lambda_2)$ is the joint prior distribution of λ_1 and λ_2, and $p(x, y \mid \lambda_1, \lambda_2)$ is the likelihood of x and y.

Now, by independence of $\lambda_1 \sim$ gamma(a_1, b_1) and $\lambda_2 \sim$gamma(a_2, b_2), we have

$$\begin{aligned} p(\lambda_1, \lambda_2) &= p(\lambda_1)p(\lambda_2) \\ &= \frac{b_1^{a_1} \lambda_1^{a_1 - 1} \exp(-b_1 \lambda_1)}{\Gamma(a_1)} \frac{b_2^{a_2} \lambda_2^{a_2 - 1} \exp(-b_2 \lambda_1)}{\Gamma(a_2)}. \end{aligned}$$

By the independence of the two samples x and y and the independence of $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ as well

$$\begin{aligned} p(x, y \mid \lambda_1, \lambda_2) &= \prod_{i=1}^{m} \lambda_1 \exp(-\lambda_1 x_i) \prod_{i=1}^{n} \lambda_2 \exp(-\lambda_1 y_i) \\ &= \lambda_1^m \exp(-\lambda_1 \sum_{i=1}^{m} x_i) \lambda_2^n \exp(-\lambda_2 \sum_{i=1}^{n} y_i). \end{aligned}$$

Hence, the joint posterior, $p(\lambda_1, \lambda_2 \mid x, y)$ of λ_1 and λ_2 is

$$\begin{aligned} p(\lambda_1, \lambda_2 \mid x, y) &\propto \lambda_1^{a_1 - 1} \exp(-b_1 \lambda_1) \lambda_2^{a_2 - 1} \exp(-b_2 \lambda_1) \lambda_1^m \exp(-\lambda_1 \sum_{i=1}^{m} x_i) \lambda_2^n \exp(-\lambda_2 \sum_{i=1}^{n} y_i) \\ &= \lambda_1^{a_1 + m - 1} \exp(-(b_1 + \sum_{i=1}^{m} x_i)) \lambda_1) \lambda_2^{a_2 + n - 1} \exp(-(b_2 + \sum_{i=1}^{n} y_i)) \lambda_2). \end{aligned}$$

Thus, $p(\lambda_1, \lambda_2 \mid x, y) = p(\lambda_1 \mid x) p(\lambda_2 \mid y)$, where $p(\lambda_1 \mid x) \sim$ gamma($a_1 + m, b_1 + \sum_{i=1}^{m} x_i$) and $p(\lambda_2 \mid y) \sim$ gamma($a_2 + n, b_2 + \sum_{i=1}^{n} y_i$), and λ_1 and λ_2 are independent under the posterior.

2. To estimate $d = \frac{\lambda_1}{\lambda_2}$ (which is random) by simulations we would generate an iid sample $\Lambda_{i1}, i = 1, \ldots, M$ from gamma$(a_1 + m, b_1 + \sum_{i=1}^{m} x_i)$ and an iid sample $\Lambda_{i2}, i = 1, \ldots, M$ from gamma$(a_2 + n, b_2 + \sum_{i=1}^{n} y_i)$. Then $(\Lambda_{i1}, \Lambda_{i2})$, $i = 1, \ldots, M$ is an iid sample of (λ_1, λ_2). Let $d_i = \Lambda_{i1}/\Lambda_{i2}, i = 1, \ldots, M$. Then, $d_i, i = 1, \ldots, M$ is an iid sample from d. Hence, we can estimate d by the sample mean of $d_i, i = 1, \ldots, M$. That is, $\hat{d} = \frac{1}{M} \sum_{i=1}^{M} d_i$.