2.

(a) Since the holding times are distributed Exp(2), we should take all the diagonal entries to be -2. Now we need the other entries chosen to make the row sums all 0 and to satisfy $\frac{g_{ij}}{2} = p_{ij}$ (where p_{ij} is the ij entry of P, that is the transition probability in the discrete-time chain). We get:

$$G = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 2/3 & -2 & 2/3 & 2/3 \\ 1/2 & 1 & -2 & 1/2 \\ 1 & 2/3 & 1/3 & -2 \end{pmatrix}$$

(b) This time we need the holding time in state i to be Exp(i), we should take all the diagonal entry $g_{ii} = -i$. As before we need the other entries chosen to make the row sums all 0 and to satisfy $\frac{g_{ij}}{i} = p_{ij}$ (where p_{ij} is the ij entry of P, that is the transition probability in the discrete-time chain). We get:

$$G = \begin{pmatrix} -1 & 1/2 & 1/2 & 0 \\ 2/3 & -2 & 2/3 & 2/3 \\ 3/4 & 3/2 & -3 & 3/4 \\ 2 & 4/3 & 2/3 & -4 \end{pmatrix}$$

3. Suppose that all the emails I receive have 0, 1, 2 or 3 attachments. Emails with k attachments arrive as a Poisson process of rate α_k. Let $A(t)$ be the number of attachments in all emails I receive in the time interval $[0, t]$.

(a) Let's think how $A(t)$ can increase in a short interval $[t, t+h]$. In other words what is $\mathbb{P}(A(t+h) = m + k \mid A(t) = m)$. We can have:

- $k = 1$: An email with one attachment arrives. This happens with probability $\alpha_1 h + o(h)$.
- $k = 2$: An email with two attachments arrives. This happens with probability $\alpha_2 h + o(h)$.
- $k = 3$: An email with three attachments arrives. This happens with probability $\alpha_3 h + o(h)$.

Note that having $k = 2$ because two emails each with one attachment arrive is possible but has probability $o(h)$. This gives

$$g_{i,i+1} = \alpha_1, \quad g_{i,i+2} = \alpha_2, \quad g_{i,i+3} = \alpha_3$$

These are the only transitions that happen with non-negligible probability.

(b) We need to set $g_{ii} = -(\alpha_1 + \alpha_2 + \alpha_3)$ to make the row sums of the generator 0. We get

$$G = \begin{pmatrix} -(\alpha_1+\alpha_2+\alpha_3) & \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & \cdots \\ 0 & -(\alpha_1+\alpha_2+\alpha_3) & \alpha_1 & \alpha_2 & \alpha_3 & 0 & \cdots \\ 0 & 0 & -(\alpha_1+\alpha_2+\alpha_3) & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ & & \ddots & \ddots & \ddots & \cdots & \end{pmatrix}$$

4. Let $Q(t)$ be the total number of people in the system (waiting and being served) at time t

(a) Just as for the $M(\lambda)/M(\mu)/1$ queue we have

$$\mathbb{P}(\text{one arrival in } [t, t+h]) = \lambda h + o(h)$$

as arrivals form a Poisson process of rate λ. So the birth parameter $\lambda_i = \lambda$ for all $i \geqslant 0$.

For the departures we need to distinguish between the cases $Q(t) = 1$ (when only one server is busy) and $Q(t) \geqslant 2$ (when both servers are busy). For the first of these cases:

$$\mathbb{P}(Q(t+h) = 0 \mid Q(t) = 1) = \mathbb{P}(\text{service ends in } [t, t+h]) + o(h) = 1 - e^{-\mu h} + o(h) = \mu h + o(h)$$

and for the second case when $m \geqslant 2$ we have:

$$\begin{aligned} \mathbb{P}(Q(t+h) = m-1 \mid Q(t) = m) &= \mathbb{P}(\text{one of the two services ends in } [t, t+h]) + o(h) \\ &= 2(1 - e^{-\mu h})(e^{-\mu h}) \\ &= 2\mu h + o(h) \end{aligned}$$

All other transitions between states have probability $o(h)$ so this is a birth-death process with parameters

$$\lambda_i = \lambda, \quad \mu_i = \begin{cases} 0 & \text{if } i = 0; \\ \mu & \text{if } i = 1; \\ 2\mu & \text{if } i \geqslant 2. \end{cases}$$

(which can also be represented by the generator

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \dots \\ \mu & -\lambda-\mu & \lambda & 0 & 0 & 0 & \dots \\ 0 & 2\mu & -\lambda-2\mu & \lambda & 0 & 0 & \dots \\ 0 & 0 & 2\mu & -\lambda-2\mu & \lambda & 0 & \dots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \end{pmatrix}$$

(b) The forwards equations are:

$$\begin{aligned} p'_{i,0}(t) &= -\lambda p_{i,0}(t) + \mu p_{i,1}(t) \\ p'_{i,1}(t) &= \lambda p_{i,0}(t) - (\mu+\lambda)p_{i,1}(t) + 2\mu p_{i,2}(t) \\ p'_{i,j}(t) &= \lambda p_{i,j-1}(t) - (2\mu+\lambda)p_{i,j}(t) + 2\mu p_{i,j+1}(t) \quad \text{for } j \geqslant 2 \end{aligned}$$

and the backwards equations are:

$$\begin{aligned} p'_{0,j}(t) &= -\lambda p_{0,j}(t) + \lambda p_{1,j}(t) \\ p'_{1,j}(t) &= \mu p_{i-1,j}(t) - (\mu+\lambda)p_{i,j}(t) + \lambda p_{i+1,j}(t) \\ p'_{i,j}(t) &= 2\mu p_{i-1,j}(t) - (2\mu+\lambda)p_{i,j}(t) + \lambda p_{i+1,j}(t) \quad \text{for } i \geqslant 2. \end{aligned}$$

(c) Letting $t \to \infty$, and assuming that $p_{i,j}(t) \to w_j$ for all i, the backward equations become:

$$\begin{aligned} \lim_{t\to\infty} p'_{0,j}(t) &= -\lambda w_j + \lambda w_j = 0 \\ \lim_{t\to\infty} p'_{1,j}(t) &= \mu w_j - (\mu+\lambda)w_j + \lambda w_j = 0 \\ \lim_{t\to\infty} p'_{i,j}(t) &= 2\mu w_j - (2\mu+\lambda)w_j + \lambda w_j = 0 \quad \text{for } i \geqslant 2. \end{aligned}$$

and the fowards equations become:

$$\begin{aligned} \lim_{t\to\infty} p'_{0,j}(t) &= -\lambda w_0 + \mu w_1 \\ \lim_{t\to\infty} p'_{1,j}(t) &= \lambda w_0 - (\mu+\lambda)w_1 + 2\mu w_2 \\ \lim_{t\to\infty} p'_{i,j}(t) &= \lambda w_{j-1} - (2\mu+\lambda)w_j + 2\mu w_{j+1} \quad \text{for } j \geqslant 2. \end{aligned}$$

So using the result of the limit of the backwards equations in the limit of the forwards equations:

$$\begin{aligned} -\lambda w_0 + \mu w_1 &= 0 \\ \lambda w_0 - (\mu+\lambda)w_1 + 2\mu w_2 &= 0 \\ \lambda w_{j-1} - (2\mu+\lambda)w_j + 2\mu w_{j+1} &= 0 \quad \text{for } j \geqslant 2. \end{aligned}$$

so we have equations:

$$
\begin{aligned}
w_1 &= \frac{\lambda}{\mu} w_0 \\
w_2 &= \frac{\mu+\lambda}{2\mu} w_1 - \frac{\lambda}{2\mu} w_0 \\
w_{j+1} &= \frac{2\mu+\lambda}{2\mu} w_j - \frac{\lambda}{2\mu} w_{j-1} \quad \text{for } j \geqslant 2.
\end{aligned}
$$

(d)
- From the first equation we have $w_1 = \frac{\lambda}{\mu} w_0 = 2\rho w_0$ as required
- Substituting for w_1 in the second equation we get

$$w_2 = \frac{\mu+\lambda}{2\mu}\frac{\lambda}{\mu} w_0 - \frac{\lambda}{2\mu} w_0 = \frac{\lambda^2}{2\mu^2} w_0 = 2\rho^2 w_0$$

  as required. So by induction we have $w_j = 2\rho^j w_0$ for all $j \geqslant 1$.
- Suppose that $w_i = 2\rho^i w_0$ for all $1 \leqslant i \leqslant j$ then from the equation for w_{j+1} we have

$$
\begin{aligned}
w_{j+1} &= \frac{2\mu+\lambda}{2\mu} 2\rho^j w_0 - \frac{\lambda}{2\mu} 2\rho^{j-1} w_0 \\
&= 2\rho^j w_0 + 2\frac{\lambda}{2\mu}\rho^j w_0 - 2\rho^j w_0 \\
&= 2\rho^{j+1} w_0
\end{aligned}
$$

  as required. So by induction we have $w_j = 2\rho^j w_0$ for all $j \geqslant 1$.

(e) For there to be a limiting distribution we need to be able to choose w_0 so that $\sum_{j\geqslant 0} w_j = 1$. That is we need

$$w_0 \left(1 + \sum_{j\geqslant 1} 2\rho^j\right) = 1$$

We can do this by setting

$$w_0 = \left(1 + \sum_{j\geqslant 1} 2\rho^j\right)^{-1}$$

provided that the sum converges. This happens if and only if $\rho < 1$ that is $\lambda < 2\mu$.

(f) If $\lambda < 2\mu$ we have $\sum_{j\geqslant 1} 2\rho^j = \frac{2\rho}{1-\rho}$. So we take

$$w_0 = \left(1 + \frac{2\rho}{1-\rho}\right)^{-1} = \frac{1-\rho}{1+\rho}.$$

So the limiting distribution is

$$w_j = \begin{cases} \frac{1-\rho}{1+\rho} & \text{if } j = 0; \\ \frac{1-\rho}{1+\rho} 2\rho^j & \text{if } j \geqslant 1. \end{cases}$$

5.

(a) We assumed that arrivals form a Poisson process. In this example they will not. The probability of an arrival in the interval $[t, t+h]$ will depend on $Q(t)$.

(b) The only thing that changes is the probability of an arrival happening. we have:

$$\begin{aligned} \mathbb{P}(Q(t+h) = n+1 \mid Q(t) = n) &= \mathbb{P}(\text{one arrival in } [t, t+h]) \cdot \mathbb{P}(\text{they do not leave}) + o(h) \\ &= (\lambda h + o(h)) p_n + o(h) \\ &= p_n \lambda h + o(h) \end{aligned}$$

So we have a birth-death process with parameters $\lambda_i = p_i \lambda$ (for $i \geqslant 0$), $\mu_i = \mu$ (for $i \geqslant 1$).

(c) We can achieve this by setting $p_i = 1$ for $0 \leqslant i \leqslant k-1$ and $p_i = 0$ for $i \geqslant k$.

6. Of course your answer will look different because your G will be different. However whichever G you chose, the general idea will be the same (see part (d) for the reason).

(a) There are lots of possible examples. I will go with:

$$G = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 2 & 2 & -4 \end{pmatrix}$$

Any 3×3 matrix with negative values on the diagonal, positive values everywhere else and row sums being 0 will do.

(b) The backwards equations are

$$\begin{pmatrix} p'_{11}(t) & p'_{12}(t) & p'_{13}(t) \\ p'_{21}(t) & p'_{22}(t) & p'_{23}(t) \\ p'_{31}(t) & p'_{32}(t) & p'_{33}(t) \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) \end{pmatrix}$$

By the assumption of the question, letting $t \to \infty$ we get that $p_{ij}(t) \to w_j$. So

$$\begin{aligned} \lim_{t\to\infty} \begin{pmatrix} p'_{11}(t) & p'_{12}(t) & p'_{13}(t) \\ p'_{21}(t) & p'_{22}(t) & p'_{23}(t) \\ p'_{31}(t) & p'_{32}(t) & p'_{33}(t) \end{pmatrix} &= \begin{pmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} w_1 & w_2 & w_3 \\ w_1 & w_2 & w_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \\ &= \begin{pmatrix} -2w_1 + w_1 + w_1 & -2w_2 + w_2 + w_2 & -2w_3 + w_3 + w_3 \\ 2w_1 - 3w_1 + w_1 & 2w_2 - 3w_2 + w_2 & 2w_3 - 3w_3 + w_3 \\ 2w_1 + 2w_1 - 4w_1 & 2w_2 + 2w_2 - 4w_2 & 2w_3 + 2w_3 - 4w_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

The 9 entries of this matrix identity together show that

$$\lim_{t\to\infty} p'_{i,j}(t) = 0$$

for all $i, j \in \{1, 2, 3\}$.

(c) The forwards equations are

$$\begin{pmatrix} p'_{11}(t) & p'_{12}(t) & p'_{13}(t) \\ p'_{21}(t) & p'_{22}(t) & p'_{23}(t) \\ p'_{31}(t) & p'_{32}(t) & p'_{33}(t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) & p_{13}(t) \\ p_{21}(t) & p_{22}(t) & p_{23}(t) \\ p_{31}(t) & p_{32}(t) & p_{33}(t) \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 2 & 2 & -4 \end{pmatrix}$$

Letting $t \to \infty$ and using the result of part (b) we get that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \\ w_1 & w_2 & w_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 2 & 2 & -4 \end{pmatrix}$$

Since each row of the righthand side is equal, this is equivalent to

$$\begin{pmatrix} 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 2 & -3 & 1 \\ 2 & 2 & -4 \end{pmatrix}$$

This gives equations

$$\begin{aligned} -2w_1 + 2w_2 + 2w_3 &= 0 \\ w_1 - 3w_2 + 2w_3 &= 0 \\ w_1 + w_2 - 4w_3 &= 0 \end{aligned}$$

which we solve to get

$$\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} w_1 & \tfrac{3}{5}w_1 & \tfrac{2}{5}w_1 \end{pmatrix}$$

Finally, since $\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}$ is a probability distribution we must have $w_1 + w_2 + w_3 = 1$ and so

$$\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} \tfrac{1}{2} & \tfrac{3}{10} & \tfrac{1}{5} \end{pmatrix}$$

(d) Why did everything magically cancel out to give the all 0 matrix in part (b)? If you look carefully, each entry is some w_i multiplied by the sum of the entries in row j of the generator. Because the rows in the generator always sum to 0, this is 0. It follows that the result of part (b) holds for any $n \times n$ generator.

Now by the argument of part (c) we get that

$$\begin{pmatrix} w_1 & w_2 & \cdots & w_n \end{pmatrix} G = \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}$$

We have proved that if a continuous-time Markov chain has a limiting distribution $\mathbf{w}$ then $\mathbf{w}$ is the solution to the matrix equation $\mathbf{w}G = \mathbf{0}$.

This should remind you of Theorem 4.6 (if a discrete-time Markov chain has a limiting distribution then it is the unique solution to the matrix equation $\mathbf{w}P = \mathbf{w}$).

7.

(a) The Poisson process waits in state i for a random time with $\mathrm{Exp}(\lambda)$ distribution and them jumps up to state $i+1$. Each of these jumps changes the parity of the state. It follows that $Y(t)$ is a the continuous-time process with state space $\{0, 1\}$ and generator

$$G = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}.$$

(b) This would no longer be a Markov chain as it will not satisfy the Markov property. If $X(t) = 0$ then $Y(t) = 0$ and the time I wait for the next jump in Y is distributed $\mathrm{Exp}(\lambda_0)$. However, if $X(t) = 2$ then $Y(t) = 0$ (same as before) but the time I wait for the next jump in Y is distributed $\mathrm{Exp}(\lambda_2)$. So how the chain evolves from state 0 depends on the previous history of the process.

8.

(a) We can't apply the argument of Sheet 9 Question 1 because the arrivals do not form a Poisson process.

(b) The first arrival must arrive on a boat. The first boat comes at time 1 and is guaranteed to have a creature on it (since $p = 1$). So the time of the first arrival is the constant random variable taking the value 1 with probability 1.

(c) Again the first arrival must come on a boat but now we must wait until the first boat with a creature on it. Since this happens with probability p, the first arrival time is distributed $\mathrm{Geom}(p)$.

(d) As in part (b) the first arrival happens at time 1 but now we must wait for the next arrival which could be either a birth or the next boat. Let T_2 be the time that the population size first reaches 2. Then certainly $1 < T_2 \leqslant 2$. The probability that there is a birth in the interval $[1, 1+x]$ (with $0 < x < 1$) is $1 - e^{-\beta x}$ (since there is one individual and births to them form a Poisson process of rate β). If there is no birth in the interval $[1, 2]$ then the population reaches size at time 2 (when the second boat arrives). So the cdf of T_2 is

$$\mathbb{P}(T_2 \leqslant x) = \begin{cases} 0 & \text{if } x < 1; \\ 1 - e^{-\beta(x-1)} & \text{if } 1 \leqslant x < 2; \\ 1 & \text{if } x \geqslant 2. \end{cases}$$

This is a rather peculiar random variable it is not discrete or continuous but a mixture of the two.

Please let me know if you have any comments or corrections

Robert Johnson
r.johnson@qmul.ac.uk