Random Processes – 2023/24 **Solutions 9**

1.

(a) The island will be uninhabited at time 3 if and only if there are no arrivals in the interval $[0, 3]$. Since the arrivals form a Poisson process of rate α the probability of this is $e^{-3\alpha}$. Hence the probability that the island is inhabited after 3 months is $1 - e^{-3\alpha}$.

(b) For there to be no new creatures in the interval $[t, t + h]$ we need that there are no arrivals and that each of the existing m creatures produces no offspring. By the assumptions of the question these are independent so:

$$\begin{aligned}
\mathbb{P}(X(t+h) = m \mid X(t) = m) &= \mathbb{P}(\text{no arrivals}) \cdot \mathbb{P}(\text{no births for a single creature})^m \\
&= (1 - \alpha h + o(h))(1 - \beta h + o(h))^m \\
&= (1 - \alpha h)(1 - \beta h)^m + o(h) \\
&= (1 - \alpha h)(1 - m\beta h + o(h)) + o(h) \\
&= 1 - \alpha h - m\beta h + m\alpha\beta h^2 + o(h) \\
&= 1 - (\alpha + m\beta)h + o(h)
\end{aligned}$$

(c) For there to be one new creature in the interval $[t, t+h]$ we must have either a single arrival and no births or no arrivals and a single birth. The single birth could happen to any one of the m creatures existing at time t. So

$$\begin{aligned}
\mathbb{P}(X(t+h) = m + 1 \mid X(t) = m) &= \mathbb{P}(\text{one arrival}) \cdot \mathbb{P}(\text{no births}) + \mathbb{P}(\text{no arrivals}) \cdot \mathbb{P}(\text{one birth}) \\
&= (\alpha h + o(h))(1 - \beta h + o(h))^m \\
&+ (1 - \alpha h + o(h))m(\beta h + o(h))(1 - \beta h + o(h))^{m-1} \\
&= \alpha h + m\beta h + o(h) \\
&= (\alpha + m\beta)h + o(h)
\end{aligned}$$

(d) To have more than two new creatures one of the following must happen in the interval $[t, t + h]$.

- at least 2 arrivals
- at least 2 creatures producing offspring
- one creature producing at least 2 offspring

Each of these happens with probability $o(h)$. So $\mathbb{P}(X(t+h) \geqslant m+2 \mid X(t) = m) = o(h)$.

(e) Parts (b,c,d) show that the probabilities $\mathbb{P}(X(t+h) = n \mid X(t) = m)$ satisfy the conditions of a birth process with birth parameters $\lambda_i = \alpha + \beta i$. It was also clear from these calculations that $\mathbb{P}(X(t+s) = n \mid X(t) = m)$ does not depend on the process up to time t. That is if $a < b$ then $X(b) - X(a)$ conditioned on $X(a)$ is independent of the process up to time a. These are the conditions we need for a birth process.

2. Let $(X(t) : \ t \geqslant 0)$ be the size of a population given by a birth process with $X(0) = 0$ and birth parameters $\lambda_i = 3 + i$.

(a) For a general birth process we have

$$\begin{aligned} p_0'(t) &= -\lambda_0 p_0(t) \\ p_1'(t) &= \lambda_0 p_0(t) - \lambda_1 p_1(t) \\ p_2'(t) &= \lambda_1 p_1(t) - \lambda_2 p_2(t) \end{aligned}$$

From the questions we have $\lambda_0 = 3$, $\lambda_1 = 4$, $\lambda_2 = 5$ so the equations are:

$$\begin{aligned} p_0'(t) &= -3p_0(t) \\ p_1'(t) &= 3p_0(t) - 4p_1(t) \\ p_2'(t) &= 4p_1(t) - 5p_2(t) \end{aligned}$$

(b) The first equation has solution $p_0 = Ce^{-3t}$ and we know $p_0(0) = 1$ so $C = 1$. So the solution is

$$p_0(t) = e^{-3t}$$

Rearranging the second equation and substituting for $p_0(t)$ gives

$$\begin{aligned} p_1'(t) + 4p_1(t) &= 3e^{-3t} \\ e^{-4t}\frac{d}{dt}\left(p_1(t)e^{4t}\right) &= 3e^{-3t} \qquad \text{(rewriting lefthand side as a derivative)} \\ \frac{d}{dt}\left(p_1(t)e^{4t}\right) &= 3e^{t}. \end{aligned}$$

And so (using the initial condition $p_1(0) = 0$),

$$p_1(t)e^{4t} = \int_0^t 3e^x \, dx = 3e^t - 3.$$

So the solution is
$$p_1(t) = 3e^{-3t} - 3e^{-4t}.$$
Rearranging the third equation and substituting for $p_1(t)$ gives
$$\begin{aligned} p_2'(t) + 5p_2(t) &= 12e^{-3t} - 12e^{-4t} \\ e^{-5t}\frac{d}{dt}\left(p_2(t)e^{5t}\right) &= 12e^{-3t} - 12e^{-4t} \quad \text{(rewriting lefthand side as a derivative)} \\ \frac{d}{dt}\left(p_1(t)e^{4t}\right) &= 12e^{2t} - 12e^{t}. \end{aligned}$$
And so (using the initial condition $p_2(0) = 0$),
$$p_2(t)e^{5t} = \int_0^t 12e^{2x} - 12e^{x}\,dx = \left[6e^{2x} - 12e^{x}\right]_{x=0}^{x=t} = 6e^{2t} - 12e^{t} + 6.$$
So the solution is
$$p_2(t) = 6e^{-3t} - 12e^{-4t} + 6e^{-5t}.$$

(c) The probability that the population has size at least 3 at time 1 is
$$\mathbb{P}(X(1) \geqslant 3) = 1 - \mathbb{P}(X(1) = 0) - \mathbb{P}(X(1) = 1) - \mathbb{P}(X(1) = 2)$$
We know that $\mathbb{P}(X(1) = k) = p_k(1)$ and putting in the values for these from above gives
$$\begin{aligned} \mathbb{P}(X(1) \geqslant 3) &= 1 - p_0(1) - p_1(1) - p_2(1) \\ &= 1 - e^{-3} - 3e^{-3} + 3e^{-4} - 6e^{-3} + 12e^{-4} - 6e^{-5} \\ &= 1 - 10e^{-3} + 15e^{-4} - 6e^{-5} \approx 0.736. \end{aligned}$$

(d) We know that S_i the time between the $(i-1)$th birth and the ith birth is distributed $\text{Exp}(\lambda_{i-1})$ so has expectation $\frac{1}{\lambda_{i-1}}$. So
$$\begin{aligned} \mathbb{E}(\text{time population reaches size } 10) &= \mathbb{E}(S_1 + S_2 + \cdots + S_{10}) \\ &= \mathbb{E}(S_1) + \mathbb{E}(S_2) + \cdots + \mathbb{E}(S_{10}) \\ &= \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_9} \end{aligned}$$
(there are 10 terms in this sum because we need to wait for 10 births).

Putting in the λ_i from the question we get:
$$\mathbb{E}(\text{time population reaches size } 10) = \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{12} = 1.603$$

(e) These are precisely the parameters we worked out in question 1 with $\alpha = 3$ and $\beta = 1$. So a situation where this could arise is measuring the size of a population where we have immigration occuring at a rate 3 per unit time and each individual producing offspring at a rate 1 per unit time with all of these processes being independent.

3.

(a) The equations are

$$p_n'(t) = -n\lambda p_n(t) + (n-1)\lambda p_{n-1}(t)$$

for $n \geqslant 1$.

(b) We need to set $p_n(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1}$ and check that the equation is satisfied. Differentiating this expression using the product rule we have

$$p_n'(t) = -\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1} + e^{-\lambda t}\lambda e^{-\lambda t}(n-1)\left(1 - e^{-\lambda t}\right)^{n-2}$$

Now we just need to substitute these expressions (for $p_n(t)$ and $p_n'(t)$) into the equation and check that it works.

One way to do this is to rewrite the equation as $f(t) = 0$ where

$$f(t) = p_n'(t) + n\lambda p_n(t) - (n-1)\lambda p_{n-1}(t).$$

Let's substitute into this and check that we get 0.

$$\begin{aligned} f(t) &= -\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1} + e^{-\lambda t}\lambda e^{-\lambda t}(n-1)\left(1 - e^{-\lambda t}\right)^{n-2} + n\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1} \\ &\quad - (n-1)\lambda e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-2} \\ &= \left(1 - e^{-\lambda t}\right)^{n-2}\left(-\lambda e^{-\lambda t}(1 - e^{-\lambda t}) + (n-1)\lambda e^{-2\lambda t} + n\lambda e^{-\lambda t}(1 - e^{-\lambda t}) - (n-1)\lambda e^{-\lambda t}\right) \end{aligned}$$

The second bracket is equal to

$$\left(-\lambda e^{-\lambda t} + \lambda e^{-2\lambda t} + n\lambda e^{-2\lambda t} - \lambda e^{-2\lambda t} + n\lambda e^{-\lambda t} - n\lambda e^{-2\lambda t} - n\lambda e^{-\lambda t} + \lambda e^{-\lambda t}\right) = 0$$

and so this is indeed a solution to the equations.

(c) We have that $\mathbb{P}(X(t) = n) = p(1-p)^{n-1}$ where $p = e^{-\lambda t}$ and so $X(t) \sim \mathrm{Geom}(e^{-\lambda t})$. Hence $\mathbb{E}(X(t)) = e^{\lambda t}$.

(d) If $Y(t)$ is a Poisson process of rate λ then $Y(t) \sim \mathrm{Po}(\lambda t)$ and so $\mathbb{E}(Y(t)) = \lambda t$.

So the expected size of a linear birth process is exponential in t in contrast to linear in t for the Poisson process. The linear birth process grows much faster.

4.

(a) In each case, if we write S_i for the time between the $(i-1)$th arrival/birth and the ith arrival/birth then

$$\mathbb{E}(\text{time of } k\text{th birth}) = \mathbb{E}(S_1) + \mathbb{E}(S_2) + \cdots + \mathbb{E}(S_k)$$

In $Y(t)$ we have that $S_i \sim \text{Exp}(2)$ while in $Z(t)$ we have $S_i \sim \text{Exp}(2^{i-1})$.

Write e_k for the expectation of the time of the kth birth in process Y and f_k for the expectation of the time of the kth birth in process Z. Using the observation above we have:

$$e_1 = \frac{1}{2}; \quad e_2 = 1; \quad e_3 = \frac{3}{2}; \quad e_4 = 2.$$

$$f_1 = 1; \quad f_2 = \frac{3}{2}; \quad f_3 = \frac{7}{4}; \quad f_4 = \frac{15}{8}.$$

So $e_k < f_k$ for $k = 1, 2, 3$ but $e_k > f_k$ for $k = 4$. Also $f_k < 2$ for all k and $e_k > 2$ for all $k \geqslant 5$ so we have $e_k > f_k$ for all $k \geqslant 4$.

(b)
- In the Poisson process $Y(t)$ we have birth parameters $\lambda_i = 2$. The sum $\sum_{i=0}^{\infty} \frac{1}{\lambda_i} = \sum_{i=0}^{\infty} \frac{1}{2}$ is infinite so by Theorem 8.3 the probability of explosion is 0.
- In the birth process $Z(t)$ we have birth parameters $\lambda_i = 2^i$. The sum $\sum_{i=0}^{\infty} \frac{1}{\lambda_i} = \sum_{i=0}^{\infty} \frac{1}{2^i} = 2$ which is finite so by Theorem 8.3 the probability of explosion is 1.

Please let me know if you have any comments or corrections

Robert Johnson
r.johnson@qmul.ac.uk