Random Processes – 2023/24 Assessment Solutions 1

In the problem sheet solutions, I usually write in quite a lot of detail giving lots of explanation and links to other material from the module. Since this is an assessment I have written thes solutions in a somewhat terser style. You can think of these as the minimum needed for the marker to give full marks with no hesitation. Of course you might get away with a bit less detail. Also there are often several ways to do a question, so your answer may not look like mine but still be worth full marks. I will give brief individual feedback along with marking and you are welcome to ask about your work by email or in a learning support hour.

1.

(a)

$$\begin{pmatrix} 0.45 & 0.45 & 0.09 & 0.01 \\ 0.2 & 0.8 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) In Markov chain terms, this is asking for the expectation of the number of visits to state 1 before absorption in state 4. We can calculate this using first-step analysis (Theorem 3.2).

(c) Let $w(1) = 1$, $w(2) = w(3) = 0$ and Let $V = \sum_{t=0}^{T-1} w(X_t)$ be the number of visits to state 1. Let $b_i = \mathbb{E}(V \mid X_0 = i)$ be the expected number of visits to state 1 given that we start in state i. Clearly $b_4 = 0$ and using Theorem 3.2 (or conditioning on the first step) we get:

$$\begin{aligned} b_1 &= 1 + 0.45b_1 + 0.45b_2 + 0.09b_3 \\ b_2 &= 0.2b_1 + 0.8b_2 \\ b_3 &= 0.5b_1 \end{aligned}$$

From the second equation $0.2b_2 = 0.2b_1$ so $b_1 = b_2$.

Using this and the third equation in the first equation:

$$b_1 = 1 + \tfrac{9}{20}b_1 + \tfrac{9}{20}b_1 + \tfrac{9}{200}b_1$$

So $\frac{11}{200}b_1 = 1$ and $b_1 = \frac{200}{11}$.

(d) This says that for the lifetime of the machine up to when it explodes, the machine spends more days running smoothly than it does running with adjustments needed.

2.

(a) This Markov chain is regular as there is a 2-step path between any two states (equivalently, the square of the transition matrix has no 0 entries). So by Theorem 4.7 it has a limiting distribution. Now that we know the limiting distribution exists, Theorem 4.6 tells us that it is also the unique equilibrium distribution so we can find it by solving $\mathbf{w}P = \mathbf{w}$ (where P is the transition matrix).

(b) The matrix equation $\mathbf{w}P = \mathbf{w}$ gives:

$$\begin{aligned}
w_1 &= \tfrac{1}{2}w_1 + \tfrac{1}{7}w_2 \\
w_2 &= \tfrac{1}{2}w_1 + \tfrac{1}{3}w_3 + \tfrac{2}{3}w_4 \\
w_3 &= \tfrac{3}{7}w_2 + \tfrac{1}{3}w_3 + \tfrac{1}{6}w_4 \\
w_4 &= \tfrac{3}{7}w_2 + \tfrac{1}{3}w_3 + \tfrac{1}{6}w_4
\end{aligned}$$

From the last two equations $w_3 = w_4$. From the first equation $w_2 = \frac{7}{2}w_1$. From these and the second equation

$$\tfrac{7}{2}w_1 = \tfrac{1}{2}w_1 + w_3$$

and so $w_3 = 3w_1$. So the limiting distribution has the form

$$\begin{pmatrix} w_1 & \tfrac{7}{2}w_1 & 3w_1 & 3w_1 \end{pmatrix}$$

Since $\mathbf{w}$ must be a probability vector we have $\frac{21}{2}w_1 = 1$ so $w_1 = \frac{2}{21}$. So

$$\mathbf{w} = \begin{pmatrix} \frac{2}{21} & \frac{7}{21} & \frac{6}{21} & \frac{6}{21} \end{pmatrix} = \begin{pmatrix} \frac{2}{21} & \frac{1}{3} & \frac{2}{7} & \frac{2}{7} \end{pmatrix}$$

(c) By the definition of a limiting distribution $p_{1,1}^{(t)}$ and $p_{2,1}^{(t)}$ both tend to $w_1 = \frac{2}{21}$ as $t \to \infty$. So these two probabilities will both be very close to $\frac{2}{21}$.

(d) The chance of the economy being in recession many months in the future will be very close to $\frac{2}{21}$. Knowing the state of the economy now will not affect this prediction.

(e) If we observe the economy for a long time then we expect the proportion of months for which it is in recession to be about $\frac{2}{21}$. (This is a consequence of Theorem 4.9.)

(An alternative answer (this time using Theorem 5.3) would be: Suppose that the economy is in recession, the expectation of the number of months until it is in recession again is $\frac{21}{2}$.

3. This question is really too individual to give a model answer for; I'm interested to read what you picked out. Here are a few things that might occur as mathematical tips.

- Draw a transition graph to see what is going on.
- Do a special case first (eg $t = 1$ rather than all t).
- Make sure that you understand the definitions of all mathematical terms in the question.
- Look up relevant background material in your first and/or second year notes.
- ... many other possibilities.

Please let me know if you have any comments or corrections

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