Extra exercises.

Compute the gradient of the following quality functions:

$$E(\mathbf{x}) = \langle \|\mathbf{x}\|^2\mathbf{x}, \mathbf{y} - \mathbf{z} \rangle \quad where \quad \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{n\times 1}$$

$$E(\mathbf{x}) = \langle \mathbf{B}(\mathbf{x} - \mathbf{y}), \alpha\mathbf{C}\mathbf{x} \rangle \quad where \quad \mathbf{x}, \mathbf{y}, \in \mathbb{R}^{n\times 1}, \quad \mathbf{B}, \mathbf{C}, \in \mathbb{R}^{n\times n}, \quad \alpha \in \mathbf{R}$$

$$E(\mathbf{x}) = \langle \mathbf{D}(\mathbf{x} + \mathbf{y}), \alpha\mathbf{x} \rangle + \|\mathbf{x} - \mathbf{y}\|^2 \quad where \quad \mathbf{x}, \mathbf{y}, \in \mathbb{R}^{n\times 1}, \quad \mathbf{D} \in \mathbb{R}^{n\times n}, \quad \alpha \in \mathbf{R}$$

$$E(\mathbf{x}) = \|\mathbf{B}(\mathbf{x} - \mathbf{y})\|^2 \quad where \quad \mathbf{x}, \mathbf{y}, \in \mathbb{R}^{n\times 1}, \quad \mathbf{B} \in \mathbb{R}^{n\times n}$$

$$E(\mathbf{x}) = \langle \mathbf{B}\mathbf{y}, \mathbf{A}\mathbf{x} \rangle \quad where \quad \mathbf{x}, \mathbf{y}, \in \mathbb{R}^{n\times 1}, \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$$

$$E(\mathbf{x}) = \langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{x} \rangle \quad where \quad \mathbf{x}, \mathbf{y}, \in \mathbb{R}^{n\times 1}, \quad \mathbf{A} \in \mathbb{R}^{n\times n}$$

$$E(\mathbf{x}) = \langle \|\mathbf{x}\|^2 \mathbf{x}, \mathbf{y} - \mathbf{z} \rangle \quad \textit{where} \quad \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{n \times 1}$$

$$E(\mathbf{x}) = \|\mathbf{x}\|^2 \langle \mathbf{x}, \mathbf{y} - \mathbf{z} \rangle = \left(\sum_k x_k^2 \right) \left(\sum_i x_i (y_i - z_i) \right)$$

$$\partial_{x_p} E(\mathbf{x}) = \left(\partial_{x_p} \sum_k x_k^2 \right) \left(\sum_i x_i (y_i - z_i) \right) + \left(\sum_k x_k^2 \right) \left(\partial_{x_p} \sum_i x_i (y_i - z_i) \right)$$

$$= 2x_p \left(\sum_i x_i (y_i - z_i) \right) + \left(\sum_k x_k^2 \right) \left(y_p - z_p \right)$$

$$= 2x_p \langle \mathbf{x}, \mathbf{y} - \mathbf{z} \rangle + (y_p - z_p) \|\mathbf{x}\|^2$$

$$\nabla E(\mathbf{x}) = 2\mathbf{x} \langle \mathbf{x}, \mathbf{y} - \mathbf{z} \rangle + (\mathbf{y} - \mathbf{z}) \|\mathbf{x}\|^2$$

$$E(\mathbf{x}) = \langle \mathbf{B}(\mathbf{x}-\mathbf{y}), \alpha \mathbf{C}\mathbf{x} \rangle \quad where \quad \mathbf{x}, \mathbf{y}, \in \mathbb{R}^{n\times 1}, \ \ \mathbf{B}, \mathbf{C}, \in \mathbb{R}^{n\times n}, \ \ \alpha \in \mathbf{R}$$

$$E(\mathbf{x}) = \alpha \langle \mathbf{B}(\mathbf{x}-\mathbf{y}), \mathbf{C}\mathbf{x} \rangle = \alpha \sum_i \left[\sum_j B_{ij}(x_j - y_j) \right] \left(\sum_k C_{ik} x_k \right)$$

$$\partial_{x_p} E(\mathbf{x}) = \alpha \sum_i \partial_{x_p} \left[\sum_j B_{ij}(x_j - y_j) \right] \left(\sum_k C_{ik} x_k \right) + \alpha \sum_i \left[\sum_j B_{ij}(x_j - y_j) \right] \partial_{x_p} \left(\sum_k C_{ik} x_k \right)$$

$$= \alpha \sum_i B_{ip} \left(\sum_k C_{ik} x_k \right) + \alpha \sum_i \left[\sum_j B_{ij}(x_j - y_j) \right] C_{ip}$$

$$= \alpha \sum_i B^{\top}_{pi} \left(\sum_k C_{ik} x_k \right) + \alpha \sum_i C^{\top}_{pi} \left[\sum_j B_{ij}(x_j - y_j) \right]$$

$$\nabla E(\mathbf{x}) = \alpha \mathbf{B}^{\top} \mathbf{C}\mathbf{x} + \alpha \mathbf{C}^{\top} \mathbf{B}(\mathbf{x}-\mathbf{y})$$

$$E(\mathbf{x}) = \langle \mathbf{D}(\mathbf{x}+\mathbf{y}), \alpha\mathbf{x}\rangle + \|\mathbf{x}-\mathbf{y}\|^2 \quad \textit{where} \quad \mathbf{x}, \mathbf{y}, \in \mathbb{R}^{n\times 1}, \;\; \mathbf{D} \in \mathbb{R}^{n\times n}, \;\; \alpha \in \mathbf{R}$$

$$E(\mathbf{x}) = \alpha\langle \mathbf{D}(\mathbf{x}+\mathbf{y}), \mathbf{x}\rangle + \|\mathbf{x}-\mathbf{y}\|^2 = \alpha \sum_i x_i \sum_j D_{ij}(x_j + y_j) + \sum_i (x_i - y_i)^2$$

$$\partial_{x_p} E(\mathbf{x}) = \alpha \sum_i \partial_{x_p} x_i \sum_j D_{ij}(x_j + y_j) + \alpha \sum_i x_i \partial_{x_p} \sum_j D_{ij}(x_j + y_j) + \partial_{x_p} \sum_i (x_i - y_i)^2$$

$$= \alpha \sum_i \delta_{ip} \sum_j D_{ij}(x_j + y_j) + \alpha \sum_i x_i D_{ip} + 2(x_p - y_p)$$

$$= \alpha \sum_j D_{pj}(x_j + y_j) + \alpha \sum_i D^{\top}_{pi} x_i + 2(x_p - y_p)$$

$$\nabla E(\mathbf{x}) = \alpha\mathbf{D}(\mathbf{x}+\mathbf{y}) + \alpha\mathbf{D}^{\top}\mathbf{x} + 2(\mathbf{x}-\mathbf{y})$$

$$E(\mathbf{x}) = \|\mathbf{B}(\mathbf{x} - \mathbf{y})\|^2 \quad where \quad \mathbf{x}, \mathbf{y}, \in \mathbb{R}^{n\times 1}, \quad \mathbf{B} \in \mathbb{R}^{n\times n}$$

$$= \sum_i \left[\sum_j B_{ij}(x_j - y_j) \right]^2$$

$$\partial_{x_p} E(\mathbf{x}) = \sum_i \partial_{x_p} \left[\sum_j B_{ij}(x_j - y_j) \right]^2$$

$$\partial_{x_p} E(\mathbf{x}) = 2 \sum_i \left[\sum_j B_{ij}(x_j - y_j) \right] B_{ip} = 2 \sum_i B_{pi}^{\top} \left[\sum_j B_{ij}(x_j - y_j) \right]$$

$$\nabla E(\mathbf{x}) = 2\mathbf{B}^{\top}\mathbf{B}(\mathbf{x} - \mathbf{y})$$

$$E(\mathbf{x}) = \langle \mathbf{B}\mathbf{y}, \mathbf{A}\mathbf{x} \rangle \quad where \quad \mathbf{x}, \mathbf{y}, \in \mathbb{R}^{n\times 1}, \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$$

$$= \sum_i \left(\sum_j B_{ij} y_j \right) \left(\sum_k A_{ik} x_k \right)$$

$$\partial_{x_p} E(\mathbf{x}) = \sum_i \left(\sum_j B_{ij} y_j \right) \partial_{x_p} \left(\sum_k A_{ik} x_k \right) = \sum_i \left(\sum_j B_{ij} y_j \right) A_{ip}$$

$$= \sum_i A^\top_{pi} \left(\sum_j B_{ij} y_j \right)$$

$$\nabla E(\mathbf{x}) = \mathbf{A}^\top \mathbf{B}\mathbf{y}$$

$$E(\mathbf{x}) = \langle \mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{x} \rangle \quad where \quad \mathbf{x}, \in \mathbb{R}^{n\times 1}, \quad \mathbf{A} \in \mathbb{R}^{n\times n}$$

$$= \sum_i \left(\sum_j A_{ij} x_j \right) \left(\sum_k A_{ik} x_k \right)$$

$$\partial_{x_p} E(\mathbf{x}) = \sum_i \partial_{x_p} \left(\sum_j A_{ij} x_j \right) \left(\sum_k A_{ik} x_k \right) + \sum_i \left(\sum_j A_{ij} x_j \right) \partial_{x_p} \left(\sum_k A_{ik} x_k \right)$$

$$= \sum_i A_{pi}^{\top} \left(\sum_k A_{ik} x_k \right) + \sum_i A_{pi}^{\top} \left(\sum_j A_{ij} x_j \right)$$

$$\nabla E(\mathbf{x}) = 2\mathbf{A}^{\top}\mathbf{A}\mathbf{x}$$