Queen Mary University of London

MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 3 (Sequences 1)

1. Recall the Demon game arising from the definition of a sequence converging to 0: "Demon chooses $\epsilon > 0$, we choose $N \in \mathbb{N}$, Demon chooses $n > N$, …". Suppose the Demon chooses $\varepsilon = 1/10$ in the first round. For each of the sequences below, state, with a brief justification, a winning N for us to choose in the second round:

   (a) $(x_n)_{n=1}^{\infty}$, where $x_n = 1/n$ for all n.

   (b) $(x_n)_{n=1}^{\infty}$, where $x_n = \cos(n\pi)/n$ for all n.

   (c) $(x_n)_{n=1}^{\infty}$, where $x_n = 1/\sqrt{n}$ for all n.

   (d) $(x_n)_{n=2}^{\infty}$, where $x_n = 1/\log_2 n$ for all $n \geq 2$.

   Solution.

   (a) We could choose $N = 10$. The Demon must choose $n > N$. Then, $|x_n| = 1/n < 1/N = 1/10 = \varepsilon$, and we win since $|x_n| < \varepsilon$.

   (b) We could again choose $N = 10$. The Demon must choose $n > N$. Then, $|x_n| = |\cos(n\pi)|/n \leq 1/n < 1/N = 1/10 = \varepsilon$, and we again win since $|x_n| < \varepsilon$.

   (c) We could choose $N = 100$. The Demon must choose $n > N$. Then, $|x_n| = 1/\sqrt{n} < 1/\sqrt{N} = 1/10 = \varepsilon$, and we win since $|x_n| < \varepsilon$.

   (d) We could choose $N = 1024$. The Demon must choose $n > N$. Then, $|x_n| = 1/\log_2 n < 1/\log_2 N = 1/10 = \varepsilon$, and we win since $|x_n| < \varepsilon$.

2. As for Question 1, but now give a winning choice for the Demon in the first round, for the following sequences $(x_n)_{n=1}^{\infty}$ (this means that the sequence does *not* converge to 0). Again, briefly justify your answer.

   (a) $x_n = \frac{1}{4} - 1/n$,

   (b) $x_n = \frac{1}{4}\cos(n\pi)$, and

   (c) $x_n = \begin{cases} 1, & \text{if } n \text{ is a perfect cube;} \\ 0, & \text{otherwise.} \end{cases}$

Solution.

(a) The Demon could choose $\varepsilon = \frac{1}{8}$. We must choose some N, and the Demon chooses $n = \max\{N, 8\}$. Then $|x_n| = \frac{1}{4} - 1/n > \frac{1}{4} - 1/N > \frac{1}{4} - \frac{1}{8} = \frac{1}{8} = \varepsilon$, and the Demon wins since $|x_n| \geq \varepsilon$.

(b) The Demon could choose $\varepsilon = \frac{1}{4}$. We must choose some N, and the Demon chooses some $n > N$. Then $|x_n| = \frac{1}{4}|\cos(n\pi)| = \frac{1}{4}|\pm 1| = \frac{1}{4} = \varepsilon$, and the Demon wins since $|x_n| \geq \varepsilon$.

(c) The Demon could choose $\varepsilon = 1$. We must choose some N, and the Demon chooses some perfect cube $n > N$, e.g., $(N+1)^3$. Then $|x_n| = 1 = \varepsilon$, and the Demon wins since $|x_n| \geq \varepsilon$.

3. For each of the following sequences state whether or not it converges to zero, and prove your answer. *Your proofs should be from "first principles": you should only use the definition of convergence as given in this course (Definition 3.1).* You must not assume any other results or techniques concerning sequences from this course, or from Calculus I or II. You *are* allowed to use the facts about real numbers proved in Chapter 2, such as the Archimedean property. If it helps, you should think first of the Demon game corresponding to convergence of the sequence, and who has the winning strategy.

(a) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{3}{n}$ for all n.

(b) $(x_n)_{n=1}^{\infty}$ given by $x_n = 3$ for all n.

(c) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{5}{2n-1}$ for all n.

(d) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{3n+1}{n^2+1}$ for all n.

(e) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{1}{10}\sin(n\pi/2)$ for all n.

(f) $(x_n)_{n=1}^{\infty}$ given by $x_n = \begin{cases} 1, & \text{if } n \text{ is a power of 2;} \\ 0, & \text{otherwise.} \end{cases}$

(g) $(x_n)_{n=2}^{\infty}$ given by $x_n = 1/\log_2 n$ for all $n \geq 2$.

Solution.

(a) The sequence $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{3}{n}$ converges to zero.

Proof. Given $\varepsilon > 0$ let $N \in \mathbb{N}$ with $N > 3/\varepsilon$ (this exists by the Archimedean principle). This implies $\frac{3}{N} < \varepsilon$. Then for $n > N$ we have

$$\left|\frac{3}{n}\right| = \frac{3}{n} < \frac{3}{N} < \varepsilon. \qquad \square.$$

This is equivalent to the following winning strategy for the Demon game:

Suppose the Demon picks $\varepsilon > 0$.
We pick $N \in \mathbb{N}$ with $N \geq \frac{3}{\varepsilon}$.
Then suppose the Demon picks $n > N$.
Then $|x_n| < 3/N < \varepsilon$ so we win.

(c) The sequence $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{5}{2n-1}$ converges to zero.

Proof. Given $\varepsilon > 0$, let $N = \lceil 5/\varepsilon \rceil$ (so $N \geq 5/\varepsilon$, and $5/N \leq \varepsilon$). Now, for all $n > N$ we have $|x_n| = 5/(2n-1) < 5/(2n-n) = 5/n < 5/N \leq \varepsilon$. □

This is equivalent to the following winning strategy for the Demon game:

Suppose the demon picks $\varepsilon > 0$.
We pick $N = \lceil 5/\varepsilon \rceil$ (so $N \geq 5/\varepsilon$, so $5/N \leq \varepsilon$).
Then suppose the demon picks $n > N$.
Then $|x_n| < 5/n < 5/N \leq \varepsilon$ so we win.

(e) The sequence $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{1}{10}\sin(n\pi/2)$ does not converge to zero.

Proof. Let $\varepsilon = 1/10$. Given any natural number N, let $n = 4N+1$, and note that $n > N$. Then

$$|x_n| = \tfrac{1}{10}\sin((4N+1)\pi/2) = \tfrac{1}{10}\sin(2N\pi + \pi/2) = \tfrac{1}{10}\sin(\pi/2) = \tfrac{1}{10} \geq \varepsilon.$$

□

This proof is equivalent to the following winning strategy for the *negated* Demon game:

We pick $\varepsilon = 1/10$.
Then suppose the demon picks N.
We pick $n = 4N+1$.
Then $|x_n| = 1/10 \geq \varepsilon$ so we win (the negated game).

(f) The sequence $(x_n)_{n=1}^{\infty}$ with $x_n = 1$ when n is a power of 2, and 0 otherwise, does not converge to 0.

Proof. Let $\varepsilon = 1$. Given $N \in \mathbb{N}$, let n be some power of 2 that is greater than N, e.g., $n = 2^N$. Then $|x_n| = 1 \geq \varepsilon$, since n is a power of 2. □

Equivalent winning strategy for Demon game:

We pick $\varepsilon > 0$.
The demon picks $N \in \mathbb{N}$.
We pick $n = 2^N$ (so $n > N$ and $x_n = 1$).
Now $|x_n| = 1 \geq \varepsilon$, so we win.

(g) The sequence $(x_n)_{n=2}^{\infty}$ given by $x_n = 1/\log_2 n$ converges to zero.

Proof. Given $\varepsilon > 0$, let $N = \lceil 2^{1/\varepsilon} \rceil$ (so $N \geq 2^{1/\varepsilon}$ and hence $\log_2 N \geq 1/\varepsilon$). Now for all $n > N$ we have $|x_n| = 1/\log_2 n < 1/\log_2 N \leq \varepsilon$. □

Equivalent winning strategy for Demon game:

Suppose the demon picks $\varepsilon > 0$.
We pick $N = \lceil 2^{1/\varepsilon} \rceil$ (so $N \geq 2^{1/\varepsilon}$ and hence $\log_2 N \geq 1/\varepsilon$. Then suppose the demon picks $n > N$.
Now $|x_n| = 1/\log_2 n < 1/\log_2 N \leq \varepsilon$, so we win.

4. Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be two sequences which converge to zero and let $(z_n)_{n=1}^\infty$ be a sequence which does not converge to zero. We define the new sequences $(\tilde{x}_n)_{n=1}^\infty$, $(\tilde{y}_n)_{n=1}^\infty$, and $(\tilde{z}_n)_{n=1}^\infty$ as follows

$$\tilde{x}_n = \begin{cases} 100 & \text{if } n \leq 10, \\ x_n & \text{if } n > 10, \end{cases}$$

$$\tilde{y}_n = \begin{cases} n^3 & \text{if } n \leq 1000, \\ y_n & \text{if } n > 1000. \end{cases}$$

$$\tilde{z}_n = \begin{cases} 1/n & \text{if } n \leq 10^{10}, \\ z_n & \text{if } n > 10^{10}. \end{cases}$$

Prove that $(\tilde{x}_n)_{n=1}^\infty$ and $(\tilde{y}_n)_{n=1}^\infty$ converge to zero and that $(\tilde{z}_n)_{n=1}^\infty$ does not converge to zero.

Solution. If you could not solve this, study the solution to the first part and then try to go back and solve the other parts in a similar fashion!

As (x_n) converges to zero, we know (by definition) that for every $\varepsilon > 0$ there exists some $N_x \in \mathbb{N}$ such that $\forall n > N_x \ : \ |x_n| < \varepsilon$. We now claim that $(\tilde{x}_n)$ converges to zero.

Proof. Given $\varepsilon > 0$, let $N = \max\{N_x, 10\}$. Then $\forall n > N$ we have $n > 10$ so $\tilde{x}_n = x_n$. Moreover, we have $n > N_x$, so $|\tilde{x}_n| = |x_n| < \varepsilon$. □

As (y_n) converges to zero, we know (by definition) that for every $\varepsilon > 0$ there exists some $N_y \in \mathbb{N}$ such that $\forall n > N_y \ : \ |y_n| < \varepsilon$. We now claim that $(\tilde{y}_n)$ converges to zero.

Proof. Given $\varepsilon > 0$, let $N = \max\{N_y, 1000\}$. Then $\forall n > N$ we have $n > 1000$ so $\tilde{y}_n = y_n$. Moreover, we have $n > N_y$, so $|\tilde{y}_n| = |y_n| < \varepsilon$. □

Remark. Note that we don't even see in the proof to which value we changed the first 10 (or 1000) elements of the sequence.

As (z_n) does *not* converge to zero, we know that there exists $\varepsilon > 0$ such that $\forall N \in \mathbb{N} \ \exists n > N \ : \ |z_n| \geq \varepsilon$. We now claim that $(\tilde{z}_n)$ does not converge to zero.

Proof. First, we pick the same $\varepsilon > 0$ as above. Then, given any $N \in \mathbb{N}$ (by the Demon) we set $\tilde{N} = \max\{N, 10^{10}\}$. From the fact that (y_n) does not converge to zero, if we would have been given this $\tilde{N}$ by the Demon, we could have found some $n_0 > \tilde{N}$ such that $|y_{n_0}| \geq \varepsilon$. We pick exactly this n_0. As $n_0 > \tilde{N} \geq N$, this is allowed. Moreover, as $n_0 > \tilde{N} \geq 10^{10}$, we have $\tilde{y}_{n_0} = y_{n_0}$, so $|\tilde{y}_{n_0}| = |y_{n_0}| \geq \varepsilon$. □

5. For each of the following sequences state whether or not it converges to zero, and *prove your answer*. You may use any result from the lectures, *provided you state it clearly.*

   (a) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{5n+17}{n^2+3n}$.

   (b) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{1}{n+10\cos(\pi n)}$.

   (c) $(x_n)_{n=1}^{\infty}$ given by $x_n = \frac{5n^2+17}{n^2+3n}$.

   Solution.

   (a) Observe that $x_n = y_n + z_n$, where $y_n = 5n/(n^2+3n)$ and $z_n = 17/(n^2+3n)$. Now $|y_n| = y_n < 5n/n^2 = 5/n$, and $|z_n| = z_n < 17/n^2 = 17/n$. We know that the sequence $(1/n)_{n=1}^{\infty}$ converges to 0, and we deduce from Corollary 3.7 that $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ do so to. Then $(x_n)_{n=1}^{\infty}$ converges to 0, by Theorem 3.9.

   (b) As an aside, note that the sequence is well defined. For any n the denominator is either $n-10$, or $n+10$. So the denominator is non-zero except possibly at $n = 10$. But when $n = 10$ the denominator is actually 20.

   Given $\varepsilon > 0$, let $N = \lceil 1/\varepsilon \rceil + 10$ (so $1/(N-10) \leq \varepsilon$). Then for all $n > N$ we have

   $$|x_n| = \frac{1}{n + 10\cos(\pi n)} \leq \frac{1}{n-10} < \frac{1}{N-10} \leq \varepsilon$$

   (since $-1 \leq \cos(\pi n) \leq 1$).

   (c) Note that $|x_n| = x_n > 5n^2/(n^2+3n) \geq 5n^2/(n^2+3n^2) = \frac{5}{4} > 1$. But $(1)_{n=1}^{\infty}$ does not converge to zero and hence, by Corollary 3.5, $(x_n)_{n=1}^{\infty}$ does not converge to 0 either.

6. Suppose $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are sequences, and that $(x_n)_{n=1}^{\infty}$ converges to 0 and that $(y_n)_{n=1}^{\infty}$ does not. Let $z_n = x_n y_n$ for all $n \in \mathbb{N}$. By giving two examples, show that $(z_n)_{n=1}^{\infty}$ may or may not converge to 0. (You do not need to justify your examples as thoroughly as in the previous questions.)

   Solution. Let $x_n = 1/n$ and $y_n = 1$ for all $n \in \mathbb{N}$. Note that $(x_n)_{n=1}^{\infty}$ converges to 0, and $(y_n)_{n=1}^{\infty}$ does not. Then $z_n = 1/n$ and $(z_n)_{n=1}^{\infty}$ converges to 0. Now keep x_n as before, but let $y_n = n$ for all $n \in \mathbb{N}$. Then $z_n = 1$ for all $n \in \mathbb{N}$ and $(z_n)_{n=1}^{\infty}$ does not converge to 0.

7. Suppose $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are bounded sequences. Show that the sequence $(z_n)_{n=1}^{\infty}$ is bounded, where $z_n = x_n + y_n$ for all $n \in \mathbb{N}$.

Solution. Since $(x_n)_{n=1}^{\infty}$ is bounded, there exists $M_x \in \mathbb{R}$, $M_x > 0$, such that $|x_n| < M_x$ for all $n \in \mathbb{N}$. Similarly, there exists $M_y \in \mathbb{R}$, $M_y > 0$, such that $|y_n| < M_y$ for all $n \in \mathbb{N}$. Set $M = M_x + M_y$. Then $|z_n| = |x_n + y_n| \leq |x_n| + |y_n| < M_x + M_y = M$. So $(z_n)_{n=1}^{\infty}$ is bounded.

8. For each of (a)–(d) give a sequence $(x_n)_{n=1}^{\infty}$ with the stated properties. In each case, the sequence $(y_n)_{n=1}^{\infty}$ is defined by $y_n = 2x_n^n$ for all $n \in \mathbb{N}$. Briefly explain your answers with respect to results and examples from the course.

(a) (x_n) converges to some $x \neq 0$, and (y_n) converges to some $y \neq 0$.

(b) (x_n) is bounded, but (y_n) does not converge.

(c) (x_n) does not converge (to any $x \in \mathbb{R}$), but (y_n) converges to zero.

(d) (x_n) does not converge (to any $x \in \mathbb{R}$), but (y_n) converges to $y \neq 0$.

Solution.

(a) E.g., let $x_n = 1$ for all n. The constant sequence $(1)_{n=1}^{\infty}$ converges to 1 (Example 3.25(i)). The sequence (y_n) given by $y_n = 2 \times 1^n = 2$ is also constant and converges to 2.

(b) E.g., let $x_n = 2$ for all n. The constant sequence $(2)_{n=1}^{\infty}$ is bounded above and below by 2. The geometric sequence $(2^n)_{n=1}^{\infty}$ does not converge (Theorem 3.14) and so (y_n) does not converge (Corollary 3.8).

(c) Let $x_n = \frac{1}{2}(-1)^{n-1}$. The sequence does not converge (to any $x \in \mathbb{R}$). This can be seen from first principles or, more easily, by noting that the difference between consecutive terms is $z_n = x_{n+1} - x_n = \pm 1$ and (z_n) does not converge to zero. Now note that $y_n = 2(-1)^{n(n-1)}(\frac{1}{2})^n = 2(\frac{1}{2})^n$. The geometric sequence $((\frac{1}{2})^n)_{n=1}^{\infty}$ converges to 0 (Theorem 3.14), and so does (y_n) (Lemma 3.6).

(d) Let $x_n = (-1)^{n-1}$. As before, this sequence does not converge (to any $x \in \mathbb{R}$). However, $y_n = 2(-1)^{n(n-1)} = 2$, so (y_n) is a constant series converging to 2. In contrast to the previous part, it is essential to let x_n equal $(-1)^{n-1}$ (equivalently $(-1)^{n+1}$) here and not $(-1)^n$!

9. **Challenge.** Suppose that the sequence $(x_n)_{n=1}^{\infty}$ converges to $x \neq 0$, and that $x_n \neq 0$ for all $n \in \mathbb{N}$. Prove that $(1/x_n)_{n=1}^{\infty}$ converges to $1/x$.

Solution. We remark that the condition $x_n \neq 0$ for all $n \in \mathbb{N}$ is just there to ensure that the sequence $(x_n)_{n=1}^{\infty}$ is well defined. Given $\varepsilon > 0$, let $\delta = \min\{\frac{1}{2}|x|, \frac{1}{2}\varepsilon x^2\}$. Choose $N \in \mathbb{N}$ such that $|x_n - x| < \delta$ for all $n > N$. Note that this inequality implies $|x_n| > \frac{1}{2}|x|$ and $|x_n - x| < \frac{1}{2}\varepsilon x^2$ for all $n > N$. Then

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \left|\frac{x - x_n}{x_n x}\right| = \frac{|x - x_n|}{|x_n|\,|x|} < \frac{\frac{1}{2}\varepsilon x^2}{\frac{1}{2}x^2} = \varepsilon,$$

for all $n > N$. Hence $(1/x_n)_{n=1}^{\infty}$ converges to $1/x$,

10. Show how to deduce part (iv) of Theorem 3.24 from Question 9.

Solution. Let $z_n = x_n \times y_n^{-1}$. We know from Question 5 that $(y_n^{-1})_{n=1}^{\infty}$ converges to $y^{=1}$. Then, by Theorem 3.24, part (iii), $(x_n y_n^{-1})_{n=1}^{\infty}$ converges to xy^{-1} as required.