Lemma Topological conjugacy is an equivalence relation

Proof Exercise. $\square$

Lemma If $h : X \to Y$ is a topological conjugacy between $f : X \to X$ and $g : Y \to Y$, then h is also a topological conjugacy between f^n and g^n, for all $n \in \mathbb{N}$.

Proof We can write the conjugacy equation as $f = h^{-1} \circ g \circ h$.

Then $f^n = f \circ f \circ \dots \circ f$

$$= (h^{-1} \circ g \circ h) \circ (h^{-1} \circ g \circ h) \circ \dots \circ (h^{-1} \circ g \circ h)$$

$$= h^{-1} \circ g \circ id \circ g \circ id \circ \dots \circ g \circ h$$

$$= h^{-1} \circ g \circ g \circ g \dots \circ g \circ h$$

$$= h^{-1} \circ g^n \circ h.$$ i.e. h is a top. conjugacy between f^n and g^n. $\square$

Proposition Topological conjugacy preserves orbits, periodic points, and (least) periods of orbits.

Proof Examine $\mathcal{O}_f(x_0) = \{f^n(x_0) : n \geq 0\}$ (the orbit of x_0 under f).

Then
$$h(\mathcal{O}_f(x_0)) = \{h(f^n(x_0)) : n \geq 0\}$$
$$= \{g^n(h(x_0)) : n \geq 0\}$$
$$= \mathcal{O}_g(h(x_0)) \quad \left(\begin{array}{c}\text{orbit of } y_0 = h(x_0) \\ \text{under } g\end{array}\right)$$

So we have seen that the image under h of an orbit is itself an orbit.

In particular, a periodic orbit for f is mapped by h to a periodic orbit for g.

__Exercise__: Check that the (least) period of the orbit is preserved by h. $\square$

Important: The preceding Proposition gives a way of showing that two maps f and g are not topologically conjugate:

Corollary If for some $n \in \mathbb{N}$, the map f has a point of period n, but g does not have a point of period n, then f and g are not topologically conjugate.

Example If f has a fixed point but g does not, then f and g are not top. conj.

Corollary If two maps f, g are topologically conjugate, then for every $n \in \mathbb{N}$, the number of n-cycles for f is equal to the number of n-cycles for g.

<u>Example</u> For $\mu = 1+\sqrt{8} \approx 3.83\ldots$, the logistic map $f_\mu(x) = \mu x(1-x)$ <u>is topologically conjugate</u> to $g_c(x) = x^2+c$

where $c = \frac{\mu}{2} - \left(\frac{\mu}{2}\right)^2$

see previous lecture

$$= \tfrac{1}{2}(1+\sqrt{8}) - \tfrac{1}{4}(1+\sqrt{8})^2$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\sqrt{8} - \tfrac{1}{4}(1+2\sqrt{8}+8)$$

$$= -\tfrac{7}{4}$$

It is convenient to study $g_c(x) = x^2+c$ with $c = -\frac{7}{4}$, as it can be shown that a <u>period-3 orbit</u> emerges at this value of c.

The equation $g_c^3(x) = x$ becomes

$$((x^2+c)^2+c)^2+c = x \qquad (*)$$

but fixed points of g_c satisfy this equation, i.e. x^2-x+c is a factor of $g_c^3(x)-x$

Factorising $g_c^3(x) - x$, we get that (*) becomes:

$$(x^2 - x + c)\left(\underbrace{x^6 + x^5 + (3c+1)x^4 + \ldots}\right) = 0$$

For <u>general</u> c we expect complex (non-real) roots, however, when $c = -7/4$ we find:

$$(x^2 - x - 7/4)\left(\underbrace{x^3 - \frac{x^2}{2} - \frac{9x}{4} - \frac{1}{8}}\right)^2$$

The roots of this cubic are real and give a period-3 orbit

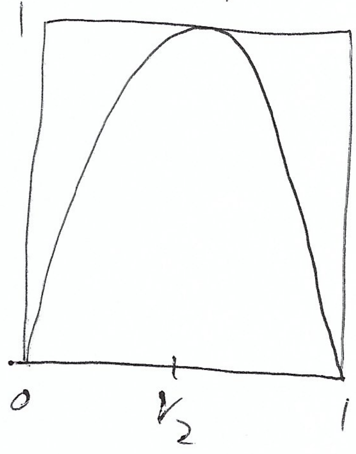
Symbolic Dynamics

Motivating discussion:

We can further analyse the specific logistic map $f_4(x) = 4x(1-x)$

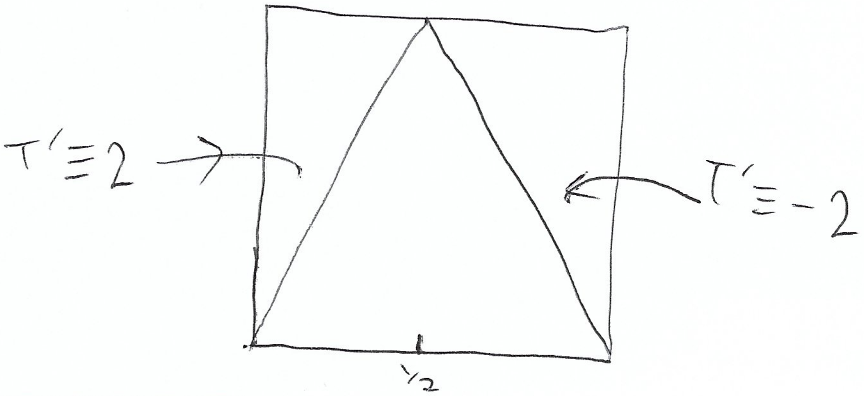
$$f_4 : [0,1] \to [0,1] \quad (\text{i.e. } \mu = 4)$$

using topological conjugacy



Defn The <u>tent map</u> T is given by

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$



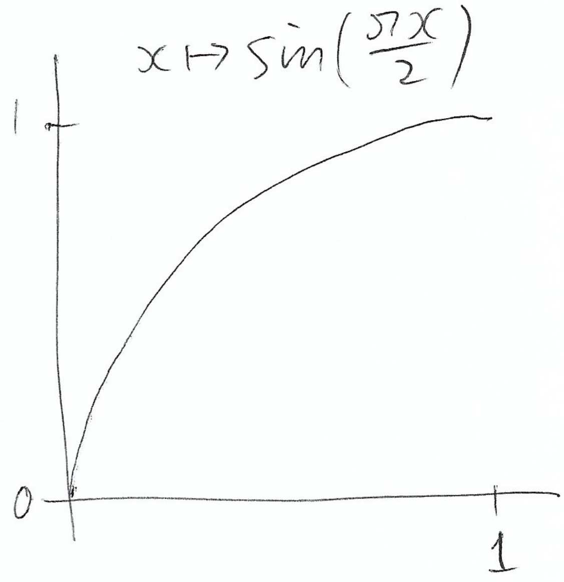
Claim (of a 'miraculous' conjugacy)

The tent map $T:[0,1]\to[0,1]$ is topologically conjugate to $f_4:[0,1]\to[0,1]$, with conjugacy map $h:[0,1]\to[0,1]$ defined by
$$h(x)=\left(\sin\left(\frac{\pi x}{2}\right)\right)^2$$
$$=\sin^2\left(\frac{\pi x}{2}\right)$$

Proof of Claim

To see this, first note that $h:[0,1]\to[0,1]$ is a homeomorphism. Note that $h(0)=0$ and $h(1)=1$, so h is surjective (by the Intermediate Value Theorem) and
$$h'(x)=2\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}>0$$
for $x\in(0,1)$, so h is injective.

So h is bijective, and h, h^{-1} are continuous



Now we need to check that the conjugacy equation $h \circ T = f_4 \circ h$ holds.

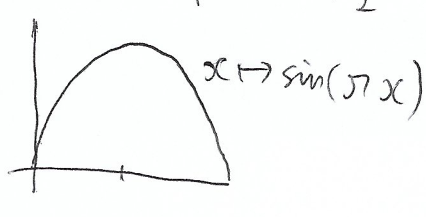
Now

$$h \circ T(x) = \begin{cases} \sin\left(\frac{\pi}{2} \cdot 2x\right)^2 & \text{if } 0 \le x < \frac{1}{2} \\ \sin\left(\frac{\pi}{2}(2-2x)\right)^2 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

$$= \begin{cases} \sin(\pi x)^2 & \text{if } 0 \le x \le \frac{1}{2} \\ \sin(\pi(1-x))^2 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

$$= (\sin(\pi x))^2$$

using symmetry of the function $x \mapsto \sin(\pi x)$ about the point $\frac{1}{2}$



Now $f_4 \circ h(x) = f_4\left(\left(\sin\left(\frac{\pi x}{2}\right)\right)^2\right)$

$$= 4\left(\sin\left(\frac{\pi x}{2}\right)\right)^2\left(1 - \left(\sin\left(\frac{\pi x}{2}\right)\right)^2\right)$$

$$= 4\left(\sin\left(\frac{\pi x}{2}\right)\right)^2\left(\cos\left(\frac{\pi x}{2}\right)\right)^2$$

$$= \left(2\sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi x}{2}\right)\right)^2$$

$$= \left(\sin(\pi x)\right)^2$$

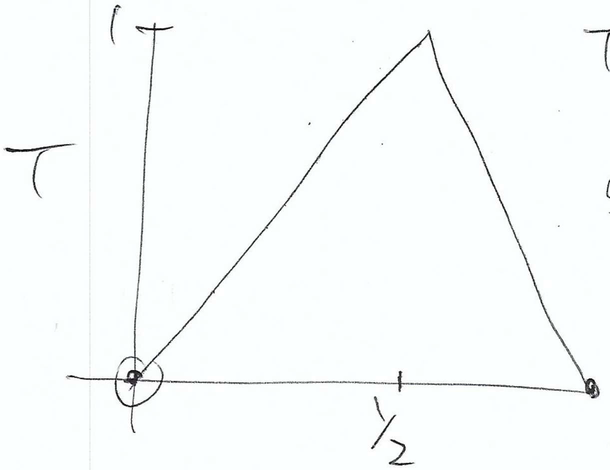
"double angle formula"

So $h(Tx) = f_4(h(x))$ for all $x \in [0,1]$,
so f_4 and T are topologically conjugate.

Let's look at periodic orbits of the tent map T (since T is piecewise-linear it is easier to work with iterates T^n, thus easier to (explicitly) find period-n points for T rather than for f_4).

<u>Question</u>: When does $T^n(x)$ equal 0 or 1? (Motivation: Since T^n is continuous, we will agree that if $T^n(a) = 0$ and $T^n(b) = 1$ then there is a fixed point of T^n in (a, b), i.e. a period-n point for T)

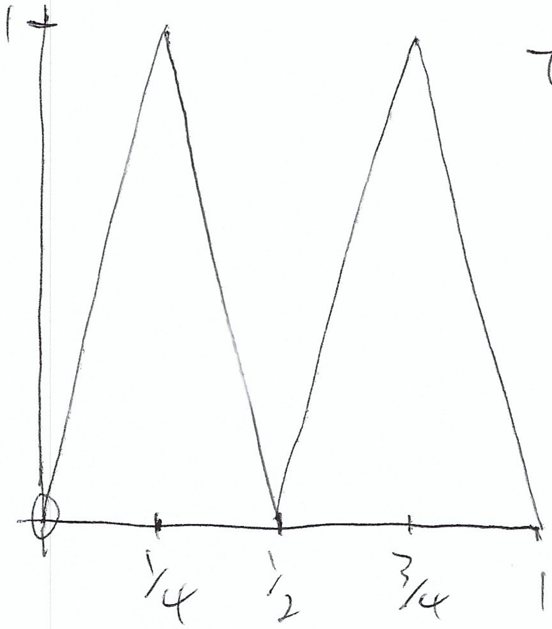
<u>$n = 1$</u>



$T(x) = 0$ when $x = 0$ or 1
<u>and</u>
$T(x) = 1$ when $x = \frac{1}{2}$

$\underline{n=2}$

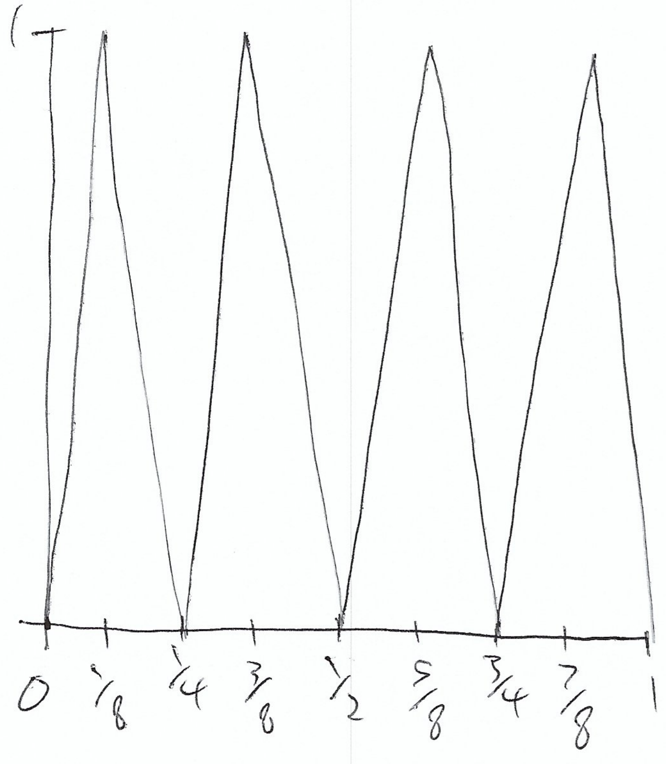
T^2



$T^2(x) = 0$ if $x = 0, \frac{1}{2}, 1$ and $T^2(x) = 1$ if $x = \frac{1}{4}, \frac{3}{4}$

$\underline{n=3}$

T^3



$T^3(x) = 0$ if $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

$T^3(x) = 1$ if $x = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$

In general $T^n(x) = 0$ when

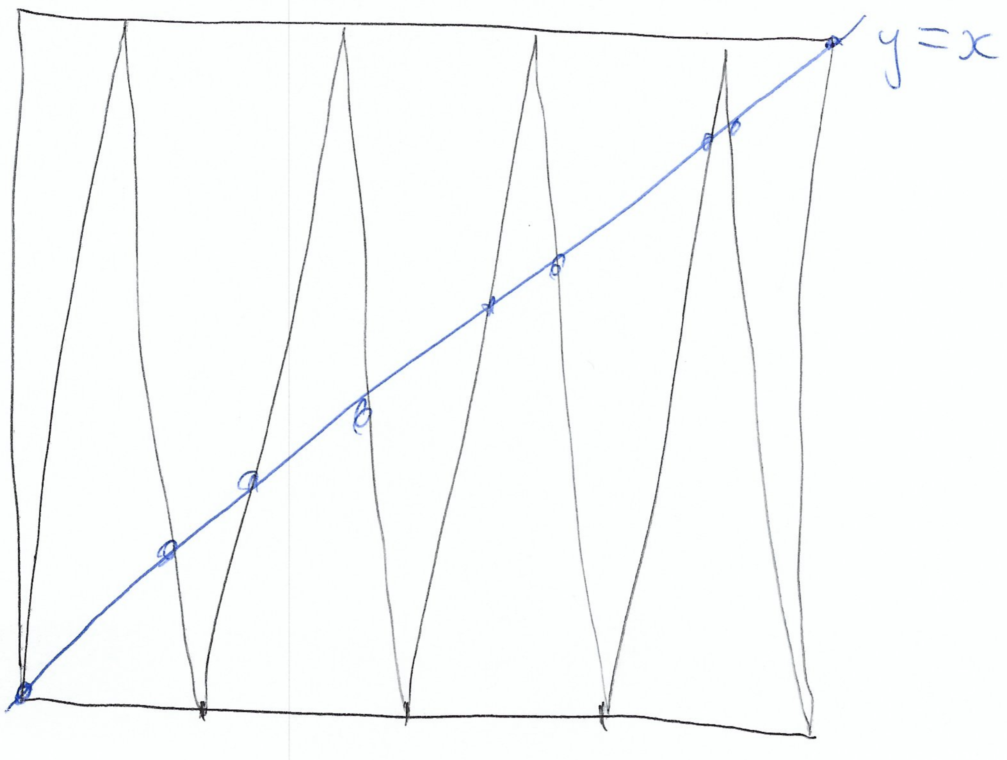
$$x = \frac{2k}{2^n} \quad \text{for } k = 0, 1, \ldots, 2^{n-1},$$

and $T^n(x) = 1$ when

$$x = \frac{2k-1}{2^n} \quad \text{for } k = 1, 2, \ldots, 2^{n-1}.$$

Note that T^n is linear (and in particular continuous) between successive points in the list $\left\{ \frac{j}{2^n} : 0 \leq j \leq 2^n \right\}$

T^n



Therefore T^n has 2^n fixed points.

So T has 2^n points of period n (note that not all of these will be of <u>least</u> period n)

Precisely one period-n point belongs to each interval of the form

$$\left[\frac{j}{2^n}, \frac{j+1}{2^n} \right].$$

<u>Examples</u> T has e.g. the following periodic points:

Fixed points: 0 and $2/3$

Period-2 points: $\{2/5, 4/5\}$ is a 2-cycle

Period-3 points: $\{2/9, 4/9, 8/9\}$ and $\{2/7, 4/7, 6/7\}$ are 3-cycles.

<u>Note</u>: To find ~~the 2-cycle~~ a point x of least period 2, we argued that if $x < \frac{1}{2}$ then the equation

$$T^2(x) = x$$

must be written as

$$T(2x) = x$$

$$2 - 2(2x) = x$$

$$2 = 5x$$

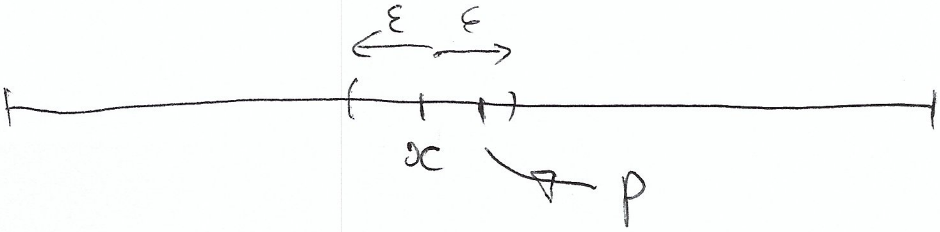
i.e. $x = \frac{2}{5}$ is of least period 2

Recall that

$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Then $T(\frac{2}{5}) = \frac{4}{5}$ is also of least period 2

<u>Remark</u> The set of <u>all</u> periodic points for T is <u>"dense"</u> in $[0,1]$, i.e. for all $x \in [0,1]$ and for all $\varepsilon > 0$, there exists a periodic point $p \in [0,1]$ such that $|x - p| < \varepsilon$



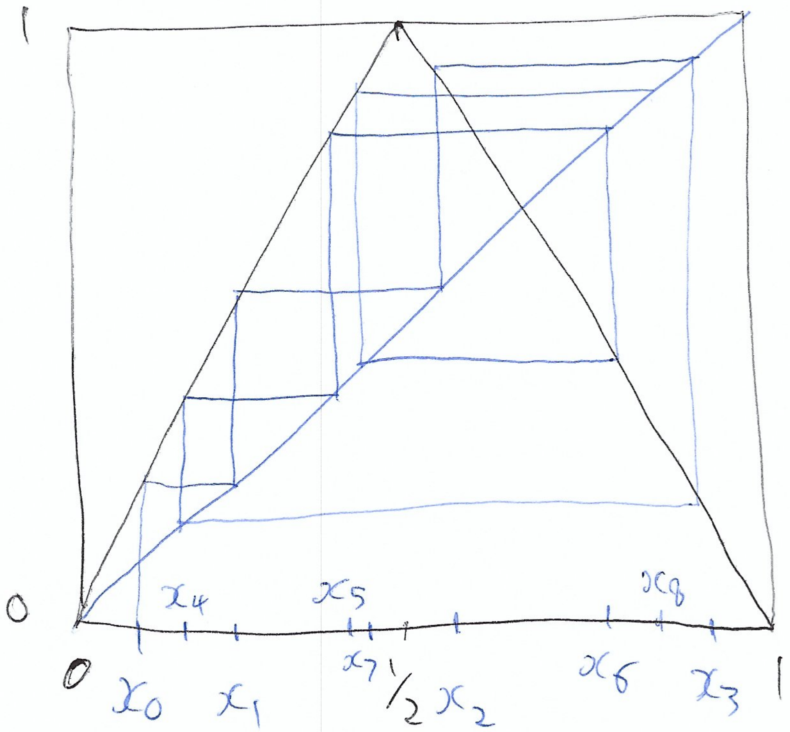
Consequently, the set of all periodic points for f_4 $(f_4(x) = 4x(1-x))$ is also dense in $[0,1]$.

Since there is a topological conjugacy between T and f_4

Symbolic Coding

We can construct a "left-right itinerary" of a point's orbit under T as follows: For each $x_0 \in [0,1]$, assign a sequence of L's and R's (L=left, R=right), i.e. a sequence where each entry / term in the sequence is either an L or a R, in such a way that the n^{th} term in the sequence is an L

if the point $x_n = T^n(x_0)$ is to the left of $\frac{1}{2}$, and is a R if $x_n = T^n(x_0)$ is to the right of $\frac{1}{2}$.



For this choice of x_0, we see that the associated sequence is

LLRRLLRLR....