Queen Mary University of London

MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 5 (Series)

1. Prove Lemma 4.6 from the lecture notes, i.e., show that if $\sum_{k=1}^{\infty} x_k = S$ and $c \in \mathbb{R}$ then $\sum_{k=1}^{\infty} cx_k = cS$.

2. (a) Which of the following sums exist? Justify your answers, using any results from the lectures/notes.

$$\text{(i)}\ \sum_{k=1}^{\infty} \frac{1}{k^3}, \qquad \text{(ii)}\ \sum_{k=1}^{\infty} \frac{1}{k2^k}, \qquad \text{(iii)}\ \sum_{k=1}^{\infty} \frac{1}{3k}.$$

   (b) Does the sum

$$\sum_{k=1}^{\infty} \left(\frac{1}{k^3} + \frac{1}{k2^k} - \frac{1}{3k} \right)$$

   exist? Prove your assertion.

3. Use the ratio test to decide which of the following series exist:

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + 3^k}{2^k + 5^k}$.

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + 5^k}{2^k + 3^k}$.

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k + 3^k + 5^k}{2^k + 3^k}$.

4. Compute the value of the following series.

   (a) $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)} = \sum_{k=1}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+2} \right)$.

   (b) $\sum_{k=1}^{\infty} \frac{2}{(k+10)(k+12)}$.

5. Prove that the series $\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$ converges for any $\alpha > 1$.

   [Hint. Try adapting the proof of Theorem 4.12, aiming this time for an *upper bound* on the partial sums.]

6. Prove or disprove the following statements:

   (a) If $\sum_{k=1}^{\infty} x_k$ converges absolutely, then $\sum_{k=1}^{\infty}(-1)^k x_k$ exists.

   (b) If $\sum_{k=1}^{\infty} x_k$ converges absolutely, then $\sum_{k=1}^{\infty} \frac{x_k}{k}$ exists.

   (c) If $\sum_{k=1}^{\infty} x_k$ converges absolutely, then $\sum_{k=1}^{\infty} k \cdot x_k$ exists.

7. In this question, $\sum_{k=1}^{\infty} x_k$ is a series that converges absolutely.

   (a) Suppose that $|x_k| \leq 1$ for all $k \in \mathbb{N}$. Prove that the series $\sum_{k=1}^{\infty} x_k^2$ converges.

   (b) Now drop the assumption that $|x_k| \leq 1$ for all $k \in \mathbb{N}$. Prove that it is still the case that the series $\sum_{k=1}^{\infty} x_k^2$ converges.

8. What happens in the previous question if we drop the word ‘absolutely’, so that $\sum_{k=1}^{\infty} x_k$ is a series that merely converges?

9. Let $x_k = \frac{2}{k(k+1)(k+2)}$ for all $k \in \mathbb{N}$ and define $S_n = \sum_{k=1}^{n} x_k$.

   (a) Evaluate S_n as a function of n.

   Hint. In a similar situation, in the notes and lectures, we used the fact that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Try something similar here, writing $\frac{2}{k(k+1)(k+2)}$ as a difference of two simpler quotients.

   (b) Evaluate the limit of the sequence $(S_n)_{n=1}^{\infty}$.

   Note that the limit from part (b) is by definition $\sum_{n=1}^{\infty} \frac{2}{k(k+1)(k+2)}$.

10. Here we study $\sum_{k=1}^{\infty} \frac{2^k}{k!}$.

    (a) Show that $k! \geq 3^{k-2}$.

    (b) Deduce that $2^k/k! \leq 4 \cdot (2/3)^{k-2}$.

    (c) Deduce that $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ exists.

11. Here we study $\sum_{k=1}^{\infty} \frac{k!}{k^k}$.

    (a) Prove that $k! \leq k^k$.

    (b) Prove that $k! \leq 2k^{k-2}$.

    (c) Deduce from (b) that $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ exists.

12. Assume that $\sum_{k=1}^{\infty} x_k$ converges and that $(y_k)_{k=1}^{\infty}$ is a bounded sequence.

(a) Find a counterexample to the statement: "$\sum_{k=1}^{\infty} x_k y_k$ converges."

(b) Prove that if we additionally assume that $x_k \geq 0$ for all $k \in \mathbb{N}$, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

13. For which values of $x \in \mathbb{R}$ do the following power series exist? Give a precise answer and justify it.

$$\text{(a)} \quad \sum_{k=1}^{\infty} 2^k x^k \qquad \text{and} \qquad \text{(b)} \quad \sum_{k=1}^{\infty} \frac{2^k x^k}{k}.$$

14. For which values of $x \in \mathbb{R}$ do the following power series exist? Give a precise answer and justify it.

$$\text{(a)} \quad \sum_{k=1}^{\infty} \frac{x^k}{k^2} \qquad \text{and} \qquad \text{(b)} \quad \sum_{k=1}^{\infty} k x^k.$$

exist?

[Hint. One possibility for (b) is to use the easily checked inequality $k \leq \alpha^{-1}(1+\alpha)^k$, valid for all $\alpha > 0$ and $k \in \mathbb{N}$.]