Week 6

What we'll cover (lecture 1)

1] Definitions: (pre) image, kernel, etc

2] Rank-Nullity theorem: statement

3] Rank-Nullity theorem: examples

I assume that you are familiar with the notion of linear map $L : V \to W$ (where V, W are real vector spaces): if $\underline{v}_1, \underline{v}_2 \in V$ and $a_1, a_2 \in \mathbb{R}$ then L satisfies $L(a_1\underline{v}_1 + a_2\underline{v}_2) = a_1 L(\underline{v}_1) + a_2 L(\underline{v}_2) \in W$.

1] $L : V \to W$ and H is a vector subspace of V
K " " " " " W

def: the preimage of K under L is

$$L^{-1}(K) = \{ \underline{v} \in V : L(\underline{v}) \in K \}$$

Th: $L^{-1}(K)$ is a vector subspace of V.

It is clear from the definition that the elements of $L^{-1}(K)$ are elements of V, so $L^{-1}(K) \subseteq V$.

We have to check closure under linear combinations

If $\underline{v}_1, \underline{v}_2 \in L^{-1}(K)$, does $a_1\underline{v}_1 + a_2\underline{v}_2 \in L^{-1}(K)$ $\forall a_i \in \mathbb{R}$? We need to check if

$L(a_1\underline{v}_1 + a_2\underline{v}_2) \in K$. Since L is linear we have

$L(a_1\underline{v}_1 + a_2\underline{v}_2) = a_1 L_1(\underline{v}_1) + a_2 L(\underline{v}_2) \in K$ since K is a vector subspace and $L(\underline{v}_i) \in K$.

- **def**: the kernel of L is the preimage of $\{\underline{0}\}$

$$\ker(L) = L^{-1}(\{\underline{0}\}) = \{\underline{v} \in V : L(\underline{v}) = \underline{0}\}$$

- **def**: The image of H under L is

$$L(H) = \{\underline{w} \in W : \underline{w} = L(\underline{v}) \text{ for some } \underline{v} \in H\}$$

Th: $L(H)$ is a vector subspace of W.

Suppose $\underline{w}_1, \underline{w}_2 \in L(H)$: is $a_1\underline{w}_1 + a_2\underline{w}_2 \in L(H)$? with $a_i \in \mathbb{R}$

$$a_1 \underline{w}_1 + a_2 \underline{w}_2 = a_1 L(\underline{v}_1) + a_2 L(\underline{v}_2) = L(a_1 \underline{v}_1 + a_2 \underline{v}_2)$$

with $\underline{v}_i \in H$. Since H is a vector subspace, then $a_1 \underline{v}_1 + a_2 \underline{v}_2 \in H$ and so $a_1 \underline{w}_1 + a_2 \underline{w}_2 \in L(H)$

- def: the range/image of L is the image of the entire space V: $\mathrm{im}(L) = L(V)$

2] Let $L: V \to W$ be a linear map

def: the rank of L is the dimension of the image of L: $\mathrm{rank}(L) = \dim(\mathrm{im}(L)) = \dim(L(V))$

def: the nullity of L is the dimension of the kernel of L: $\mathrm{nul}(L) = \dim(\ker(L)) = \dim(L^{-1}(\{\underline{0}\}))$

Th (Rank-Nullity theorem). $L: V \to W$ is a linear map between the vector spaces V and W. Then

$$\boxed{\dim(V) = \mathrm{rank}(L) + \mathrm{nul}(L)}$$

(see the typewritten notes for the optional proof)

3] Ex 1: Consider $V = P_3$, $W = P_2$ and L is $D = \frac{d}{dx}$

$$f(x) \in P_3 \to f'(x) \in P_2$$

From $\frac{df}{dx} = 0$, ie. $\frac{d}{dx}\left(a_1 + a_2 x + a_3 x^2 + a_4 x^3\right) = a_2 + 2a_3 x + 3a_4 x^2 = 0$

we see that the elements $f(x) = a_1$ form $\mathrm{Ker}(D)$

Thus $\mathrm{nul}(D) = 1$. From the same result above, we see that $\mathrm{Im}(D) = P_2$, so $\mathrm{rank}(D) = 3$. We have

$$\mathrm{rank}(D) + \mathrm{nul}(D) = 3 + 1 = 4 = \dim(P_3) \quad \checkmark$$

Ex2: $V = W = \mathbb{R}^{2\times 2}$ and L is $L(A) = A - A^T$.

A generic element is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which is mapped to

$L(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & b-c \\ -(b-c) & 0 \end{pmatrix}$. Thus we see

that $\mathrm{im}(L)$ in this case is the set of antisymmetric matrices in $\mathbb{R}^{2\times 2}$. Thus $\mathrm{rank}(L) = 1$. The kernel is the set of matrices with $b = c$ and arbitrary a, d, so it is the set of symmetric matrices in $\mathbb{R}^{2\times 2}$. Thus $\mathrm{nul}(L)$ is in this case 3 and we have

$$\mathrm{nul}(L) + \mathrm{rank}(L) = 3 + 1 = 4 = \dim\left(\mathbb{R}^{2\times 2}\right) \quad \checkmark$$

Ex3: $V = \mathbb{R}^4$, $W = \mathbb{R}^3$ and the matrix $A \in \mathbb{R}^{3\times 4}$

$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ generates a linear map from $\mathbb{R}^4$ to $\mathbb{R}^3$

Acting with A on a generic element of $\mathbb{R}^4$ we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix}$$

Thus elements of the form $\begin{pmatrix} 0 \\ 0 \\ a_3 \\ a_4 \end{pmatrix}$ are in the kernel of $L = A$. So $\mathrm{nul}(A) = 2$

The image of $L = A$ is made up of elements of $\mathbb{R}^3$ of the form $\begin{pmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{pmatrix}$ and so it is spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

Thus we have that the rank of $L = A$ is 2, in agreement with the theorem

$$\dim(V) = 4 = \mathrm{nul}(A) + \mathrm{rank}(A) = 2 + 2$$

What we'll cover (lecture 2-3)

4] Linear maps, coordinatisation, matrices

5] Row space, column space, null space for matrices (definitions)

6] Rank-nullity for matrices

4] Definition of a matrix associated to a linear map $L : U \to V$ where U, V are vector spaces

- Let $B = \{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$ be a basis for U
- Let $C = \{\underline{c}_1, \underline{c}_2, \dots, \underline{c}_m\}$ be a basis for V

The coordination map is a bijective correspondence between a vector space of dimension n and $\mathbb{R}^n$ whose definition <u>depends</u> on the choice of a basis.

Ex coord. map. for U

$$\underline{u} \in U, \quad \text{then} \quad \underline{u} = \sum_{i=1}^{n} \beta_i \underline{b}_i \qquad \beta_i \in \mathbb{R}$$

⊖ β_i are uniquely fixed given $\underline{u}$
⊖ any choice of β_i yields a $\underline{u} \in U$
⊖ any $\underline{u} \in U$ can be written as above

} bijective

$$[\ \ ]_B : U \to \mathbb{R}^n \qquad [\underline{u}]_B = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

Similarly $[\ \ ]_C : V \to \mathbb{R}^m$. For instance

$$v = \sum_{i=1}^{m} \gamma_i \underline{c}_i \in V \qquad [\underline{v}]_C = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_m \end{pmatrix}$$

Equipped with this, we define the matrix associated to

L as follows

$$[L(\underline{b}_1)]_C = \begin{pmatrix} \gamma_{11} \\ \gamma_{21} \\ \vdots \\ \gamma_{m1} \end{pmatrix}, \ \ldots\ldots, \ [L(\underline{b}_n)]_C = \begin{pmatrix} \gamma_{1n} \\ \gamma_{2n} \\ \vdots \\ \gamma_{mn} \end{pmatrix}$$

$$[L]^B_C = \begin{pmatrix} \gamma_{11} & \cdots\cdots & \gamma_{1n} \\ \gamma_{21} & & \gamma_{2n} \\ \vdots & & \vdots \\ \gamma_{m1} & & \gamma_{mn} \end{pmatrix}$$

It's the matrix whose columns are the coordinates of $L(\underline{b}_i)$ according to the basis C

Remarks: • Of course if you choose a different basis for U or V, you'll get a different associated matrix.

• $[\ ]_B : U \to \mathbb{R}^n$ is a linear map

<u>Ex 1</u>: Consider the linear map discussed in Ex 1 of 3]

$D = \frac{d}{dx} : P_3 \to P_2$ (i.e. $f(x) \in P_3 \to f'(x) \in P_2$)

Choose $\underline{b}_i = x^{i-1}$ $i = 1, \ldots, 4$ as basis for P_3

$\underline{c}_1 = x^{i-1}$ $i = 1, \ldots, 3$ as basis for P_2

We have $[D(\underline{b}_1)]^B_C = [D(1)]^B_C = 0 \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$

$$[D(\underline{b}_2)]_C^B = [D(x)]_C^B = 1 \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$[D(\underline{b}_3)]_C^B = [D(x^2)]_C^B = 2x \rightarrow \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$[D(\underline{b}_4)]_C^B = [D(x^3)]_C^B = 3x^2 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Thus we have

$$[D]_C^B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

<u>Ex 2</u>: Consider the linear map discussed in Ex 2 of 3]

As a basis for $\mathbb{R}^{2\times 2}$ we choose

$$\underline{b}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \underline{b}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \underline{b}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \underline{b}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

For $D = A - A^T$, we have

$$[D(b_1)]_B^B = 0 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad [D(b_4)]_B^B = 0 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[D(b_2)]_B^B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad [D(b_4)]_B^B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Downarrow$$

$$[D]_B^B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Comment: when $\dim(U) = \dim(V)$ then the associated matrix is square

Proposition: With this notation for $\underline{u} \in U$

matrix multiplication

$$\left[L(\underline{u})\right]_C = \left[L\right]_C^B \cdot \left[\underline{u}\right]_B$$

Proof: $\underline{u} = \sum_{i=1}^{n} \beta_i \underline{b}_i$ and

L is linear $\qquad$ $[\ ]_C$ is linear

$$\left[L(\underline{u})\right]_C = \left[L\left(\sum_{i=1}^{m} \beta_i \underline{b}_i\right)\right]_C \overset{L \text{ is linear}}{=} \left[\sum_{i=1}^{m} \beta_i L(\underline{b}_i)\right]_C \overset{[\ ]_C \text{ is linear}}{=}$$

$$\sum_{i=1}^{m} \beta_i \left[L(\underline{b}_i)\right] = \sum_{i=1}^{m} \beta_i \begin{pmatrix} \gamma_{1i} \\ \gamma_{2i} \\ \vdots \\ \gamma_{mi} \end{pmatrix} = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & & \\ \gamma_{m1} & \cdots & \gamma_{mn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} =$$

$$= \left[L(\underline{u})\right]_C^B \cdot \left[\underline{u}\right]_B \quad \checkmark$$

We can summarise the situation as follows

$$\begin{tikzcd} U \arrow[r, leftrightarrow, "{[\ ]_B}"] \arrow[d, "L"'] & \mathbb{R}^n \arrow[d, "\mathcal{L}_{[L]_C^B}"] \\ V \arrow[r, leftrightarrow, "{[\ ]_C}"] & \mathbb{R}^m \end{tikzcd} \qquad \begin{tikzcd} \underline{u} \arrow[r, leftrightarrow] \arrow[d] & {[\underline{u}]_B} \arrow[d, "{[L]_C^B \cdot [\underline{u}]_B}"] \\ L(\underline{u}) \arrow[r, leftrightarrow] & {[L(\underline{u})]_C} \end{tikzcd}$$

From the linearity of the coordinatisation $[\ ]$, one can straightforwardly prove that

$$[\alpha L]_C^B = \alpha \left[L\right]_C^B, \text{ with } \alpha \in \mathbb{R} \text{ and } L: U \to V$$

$$[L + L']_C^B = [L]_C^B + [L']_C^B, \text{ with } L, L': U \to V \quad \text{lin. maps}$$

By following very similar steps as in the proposition above, one can prove the following statement

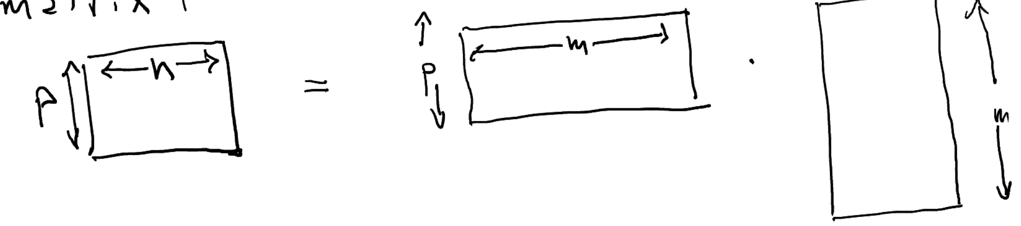
<u>Cor</u>: If B basis for U of dim. n

C " " V " " m

D " " W " " p

and if $U \xrightarrow{L} V \xrightarrow{M} W$ where L, M are linear maps, then $[M \circ L]^B_D = [M]^C_D \cdot [L]^B_C$

matrix $p \times n$



Summary: composition of linear map $\underset{\text{coord.}}{\longleftrightarrow}$ matrix multiplicat.

In pictures

$$\begin{tikzcd} U \arrow[r, leftrightarrow, "{[\ ]_B}"] \arrow[d, "L"'] & \mathbb{R}^n \arrow[d] \\ V \arrow[r, leftrightarrow, "{[\ ]_C}"] \arrow[d, "M"'] & \mathbb{R}^m \arrow[d, "\mathcal{L}_{[M]^C_D}"] \\ W \arrow[r, leftrightarrow, "{[\ ]_D}"] & \mathbb{R}^p \end{tikzcd}$$

So $\mathcal{L}_{[M]^C_D} \circ \mathcal{L}_{[L]^B_C} = \mathcal{L}_{[M]^C_D \cdot [L]^B_C}$

5] Def: The row space of a $\mathbb{R}^{m \times n}$ matrix is the vector subspace of $\mathbb{R}^n$ spanned by the rows of A

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \rightarrow \mathrm{span}\left((a_{11}, \dots, a_{1n}), \dots, (a_{m1}, \dots, a_{mn}) \right)$$

Def: the column subspace of A is the vector subspace of $\mathbb{R}^m$ spanned by the columns of A

Def: The nullspace of A is

$$N(A) = \{ \underline{x} \in \mathbb{R}^n : A\underline{x} = 0 \}$$

Def The rank of a matrix A is the dimension of the row space

Def: The nullity of a matrix A is the dimension of its nullspace

Theorem (4.70 of the notes): The dimension of the row space is the same as the dimension of the column space (so $\mathrm{rank}(A) = \dim(\mathrm{col}(A))$).

Ex 1 Consider the matrix in Ex 1 of 3]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$ row(A) is spanned by (1000) and (0100), so rank(A) = 2

This is indeed equal to dim(col(A)) since col(A) is spanned by $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$.

N(A) is spanned by $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ so nul(A) = 2

Notice that these are the same results we obtained last week when we studied A as an abstract linear map from $V = \mathbb{R}^4$ to $W = \mathbb{R}^3$.

The definitions of rank-nullity for matrices seem consistent with those for linear maps!

Ex 2] In Ex 1 of 3] we had $D = \frac{d}{dx} : P_3 \to P_2$.

rank(D) = 3, nul(D) = 1

Ex 1 of 4]

The associated matrix derived above was $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

It is clear that rank(A) = 3 and nul(A) = 1 as N(A) is spanned by $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$.

6] **Theorem**: Row equivalent matrices have the same row space and so the same rank

Proof: a matrix B is row equivalent to a matrix A iff B can be obtained by elementary row oper. They involve just linear combinations of rows. So $row(B) \subseteq row(A)$. The elementary row operations are invertible so we can obtain A from B reversing each step, so $row(A) \subseteq row(B)$. So $row(A) = row(B)$. ✓

$\Rightarrow$ Thus in order to calculate the rank of A we can

- bring A to the REF
- the non-zero rows span $row(A)$
- the number of non-zero rows is $rank(A)$

Theor: Let $A \in \mathbb{R}^{m \times n}$. We have $\boxed{rank(A) + nul(A) = n}$

Proof: Bring A to REF. Then $rank(A)$ is the number of non-trivial rows and so the number of leading variables.

Instead $nul(A)$ is the number of free variables since each free variables yields a non-trivial solution of $Ax = 0$. Since the total number of variables is n, we proved the theorem.

Examples... after reading week.