Week 7

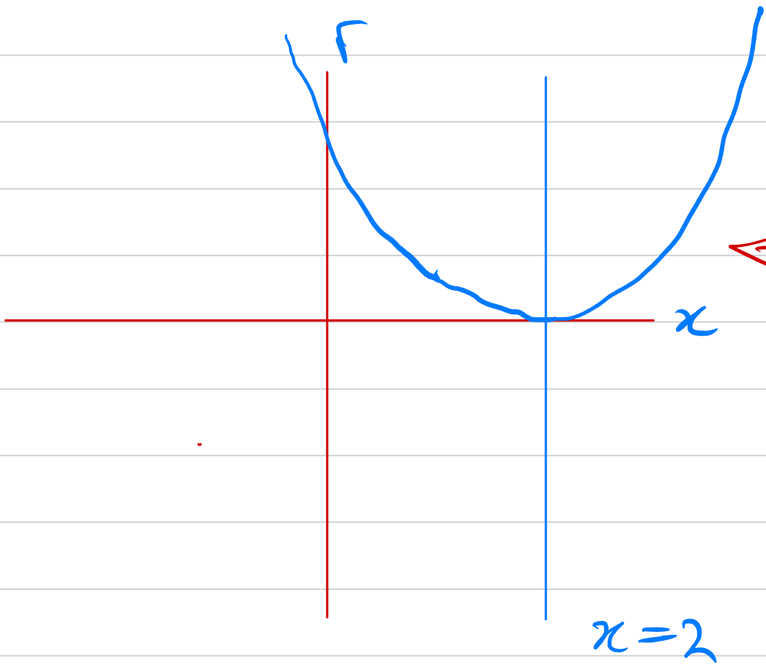
Tuesday discussion (9.00 - 10.45).

Week 7 Revision

$$\dot{x} = rx + \frac{x^3}{1+x^2}$$

Bifurcation?

Q2 exam. Draw the bifurcation diagram



Fixed pts

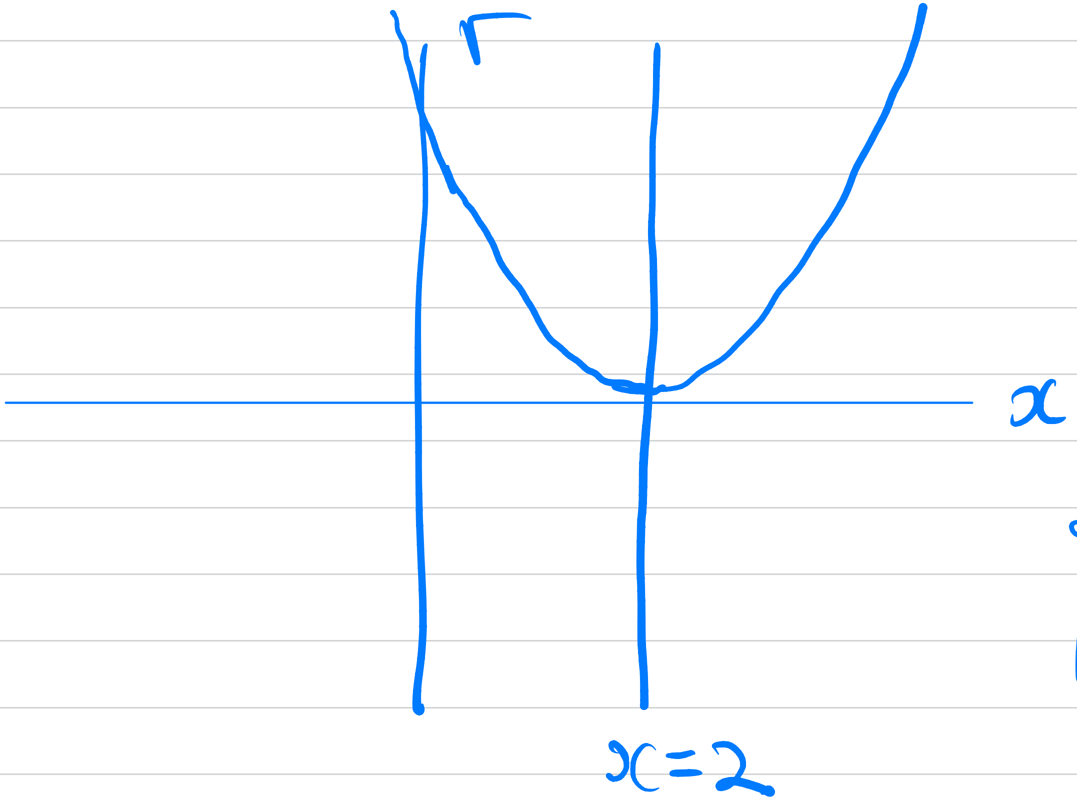
$$\dot{x} = (x-2)\left((x-2)^2 - r\right)$$

Curves $x=2$, $r = (x-2)^2$

$$\dot{x} = (x-2)\left((x-2)^2 - r\right)$$

FP set $\quad \dot{x} = 0$

$$x - 2 = 0 \quad , \quad (x-2)^2 - r = 0$$



$x = 2$

$r = (x-2)^2$

Pitchfork ?

$x = 2, r = 0$

A, B, C, E ?

$C \neq 0 \quad E \neq 0$

Strogatz 3.4.10

$$\dot{x} = rx + \frac{x^3}{1+x^2}.$$

Note $x=0$ is a fixed point for all r.

Bifurcation occurs at a change of linear stability of a fixed point as r increases

So we need $$r.x + \frac{x^3}{1+x^2} = 0$$

and $$\frac{\partial}{\partial x}\left(rx + \frac{x}{1+x^2}\right) = 0$$

Linear stability

$$\frac{\partial}{\partial x}\left(rx+\frac{x^3}{1+x^2}\right)\Bigg|_{x=0}$$

unstable if > 0

stable if < 0

$$= r + \frac{(1+x^2)\cdot 3x^2 - x^3(2x)}{(1+x^2)^2}\Bigg|_{x=0} = r$$

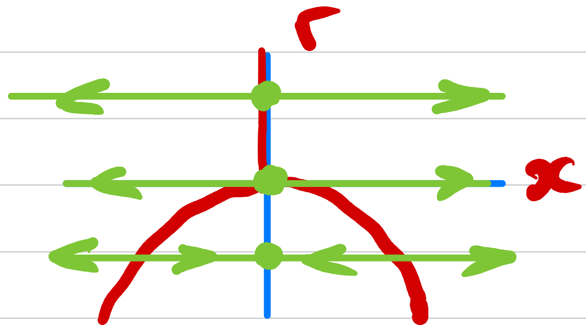
$r=0$ bifurcation?

$$\dot{x} = rx + \frac{x^3}{1+x^2} = rx + x^3 - x^5 + \dots$$

$C \neq 0 \ (=1)$ $\quad E = 1$ $\quad \dot{x} = rx + x^3$

$x(r+x^2)$

FP set $x=0$, $r+x^2=0$



Dynamical Systems on $\mathbb{R}$

① FPs ?

② Direction of flow with incr. time between fixed points.

$+ = \rightarrow$

$- = \leftarrow$

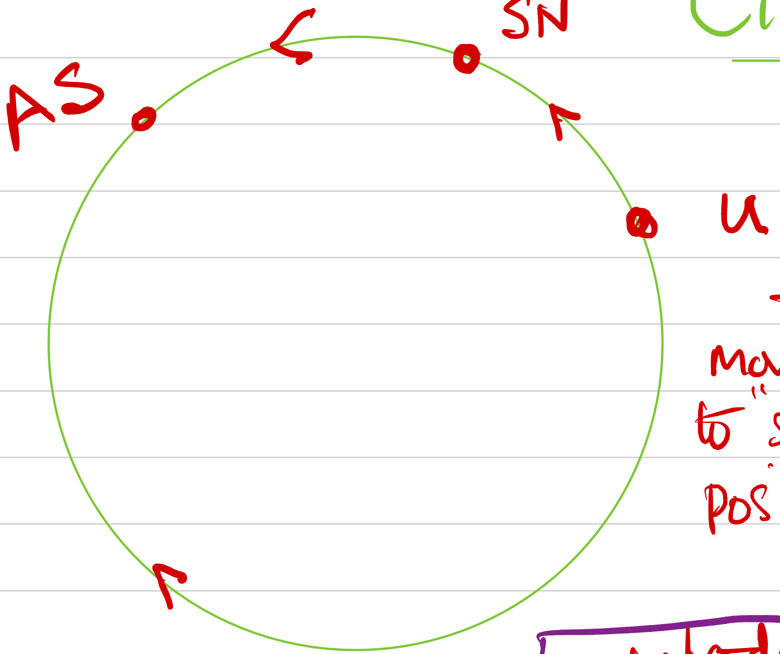
Note $\dot{x} = \dfrac{p(x)}{(1+x^2)}$. Given $1+x^2 > 0$ for all $x \in \mathbb{R}$

Zeros of $p(x)$ = zeros of $\dfrac{p(x)}{1+x^2}$

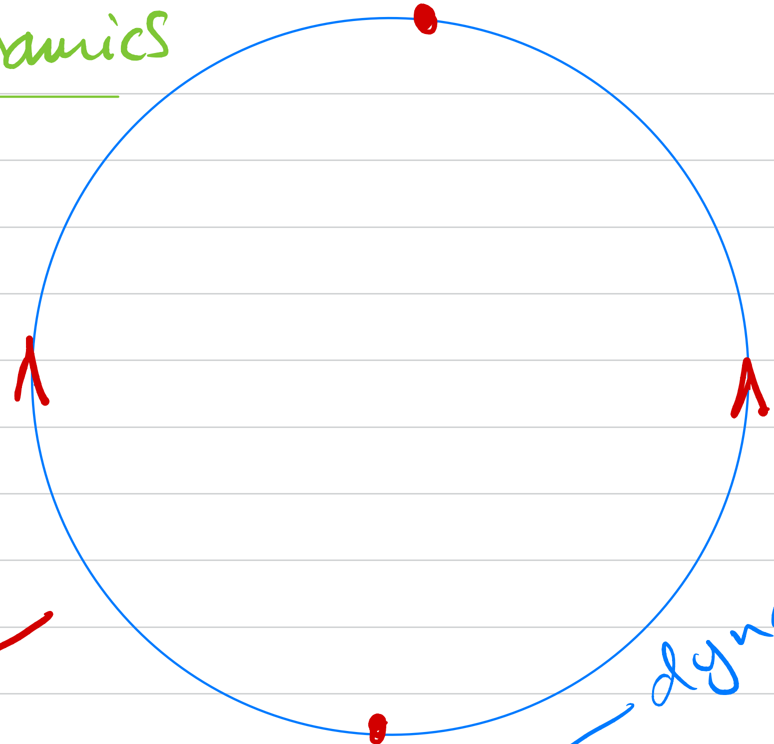
$\operatorname{sgn}(\dot{x}) = \operatorname{sgn}(p(x))$ given $1+x^2 > 0$

$\therefore$ Qualitative equivalent phase portraits for $\dfrac{p(x)}{1+x^2}$ and $p(x)$

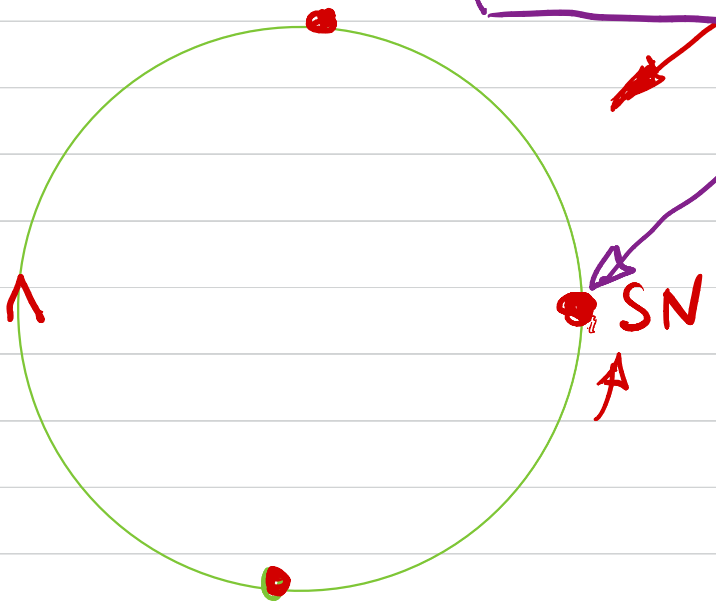
Circle dynamics



move pts to "simple" positions



introduce $x(1-\cos\theta)$



dyna.

$$\dot\theta = \cos\theta \cdot \boxed{(1-\cos\theta)}$$

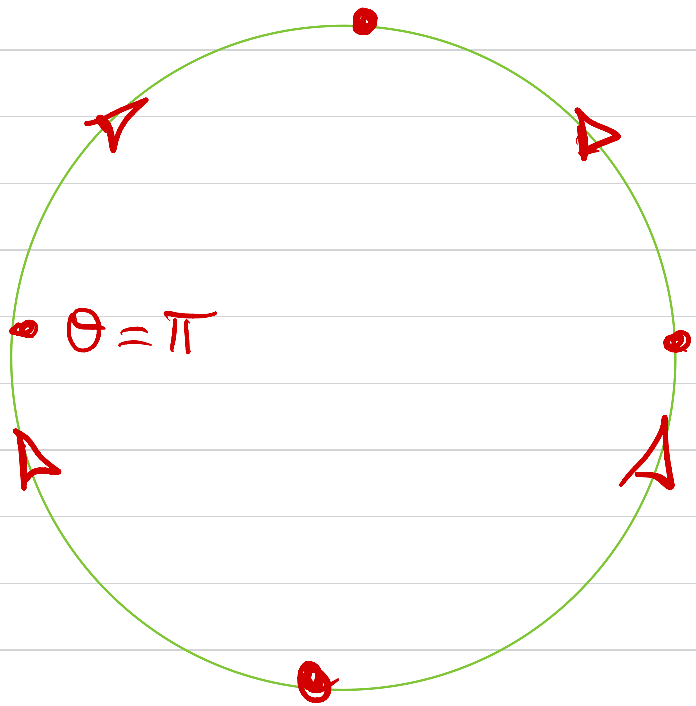
sgn(blue) = sgn(blue x green), $\theta \neq 0$

$$\dot{\theta} = \cos\theta\,(1-\cos\theta)(1+\cos\theta).$$

$+\ \phi\ -$

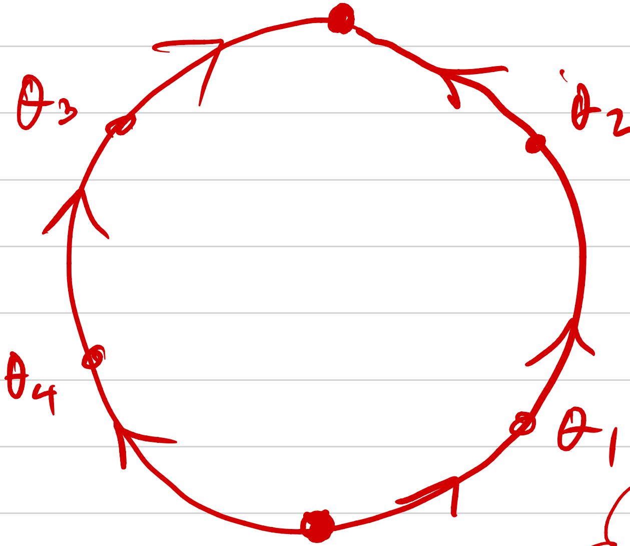
+ve except for $\theta = 0$

+ve except for $\theta = \pi$



$\theta = 0.$

$$\dot{\theta} = \cos\theta$$



$$\dot{\theta} = \cos\theta \times \prod_{i=1}^{4} \left(1 - \cos(\theta - \theta_i)\right)$$

Also note
$$1 - \cos\theta = 1 + \cos(\theta + \pi)$$

How to introduce fixed points without changing flow direction *

2D

$$\underline{\dot{x}} = \underline{A}\,\underline{x} \qquad \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1D

$$\boxed{\dot{x} = ax}$$

$$A = [a]$$

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

✓ proper node

$\lambda_1 \cdot \lambda_2 > 0$

J

λ_1, λ_2 real ($\lambda_1 = \lambda_2$ ok).

Improper node → $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ → A not diagonal & equal eigenvalues $\lambda_1 = \lambda_2 = \lambda \neq 0$

non-diagonal A

$\lambda_1, \lambda_2 = \alpha + i\beta$ $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$

Suppose $P^{-1}AP = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ $\qquad \lambda_1 = \lambda_2 = \lambda.$

$$\Rightarrow A = P\lambda I P^{-1} = \lambda P I P^{-1} = \lambda I$$

$\therefore$ if A has non-diagonal & has $\lambda_1 = \lambda_2$, then A is not reducible to $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, but $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ ✓

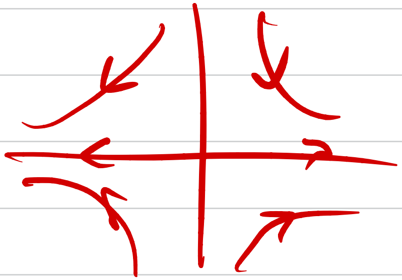
$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\lambda_1 \cdot \lambda_2 < 0$ $\therefore \frac{\lambda_2}{\lambda_1} < 0$

$$dt = \frac{dx}{\lambda_1 x} = \frac{dy}{\lambda_2 y}$$

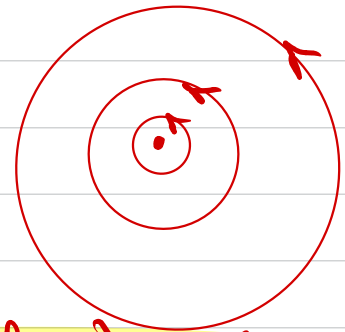
$$y = C x^{\lambda_2/\lambda_1}$$

$$y x^{-\lambda_2/\lambda_1} = C \rightsquigarrow$$ hyperbolic curves asymptotic to x & y axes.

saddles



Centre
$\alpha = 0, \beta \neq 0$



stable, but not AS.

Eigenvalues if $\lambda_1, \lambda_2 \neq 0$ determine the type of fixed pt at $\underline{0}$