Lecture 2

Exercise: Show that

$$\frac{1+\sqrt{5}}{2} = [1; 1, 1, \ldots]$$

Theorem: Every irrational number is the value of a unique continued fraction expansion.

Pf: By the previous theorem

$$r = [a; a_1, a_2, \ldots].$$

The RHS is uniquely determined:

$$a = \lfloor r \rfloor, \quad a_1 = \left\lfloor \frac{1}{r-a} \right\rfloor$$

$$\rho_1 = \frac{1}{r-a}, \quad \rho_2 = \frac{1}{\frac{1}{r-a} - a_1}$$

$$a_2 = \lfloor \rho_2 \rfloor$$

so $a, a_1, a_2 \ldots$ are all uniquely determined.

Periodic Continued fraction

Def: An infinite continued fraction is called periodic if $\exists\ l, N$ with $l > 0$ s.t. the sequence a_n ~~stabilizes~~ at the N'th repeats itself with the cycle length l:

$$a_N = a_{N+l} = a_{N+2l} = \cdots$$

$$a_{N+1} = a_{N+l+1} = a_{N+2l+1} = \cdots$$

$$\vdots$$

$$a_{N+l-1} = a_{N+2l-1} = a_{N+3l-1} = \cdots$$

Or $\quad a_{n+l} = a_n \quad \forall n \geq N.$

We can write the expansion as

$$r = [a; a_1, a_2, \ldots, a_{N-1}, \overline{a_N, \ldots, a_{N+l-1}}]$$

Def: It is called purely periodic if $N = 0$ i.e. $r = [\overline{a; a_1 \ldots a_{l-1}}]$

Example:

1) $[2; 1, 2, 1, \dots] = [2; \overline{1}]$

2) $[3; 5, 2, 1, 2, 1, \dots] = [3; 5, \overline{2, 1}]$

3) $[1; 1, 1, 1, \dots] = [\overline{1}]$

Ex: Find the value of $[2; 1, 2, 1, \dots]$

Ans: Let $x = [2; 1, 2, 1, \dots]$

$$x = 2 + \frac{1}{[1; 2, 1, \dots]}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{[2; 1, 2, 1, \dots]}}$$

$$= 2 + \frac{1}{1 + \frac{1}{x}}$$

$$\Rightarrow x - 2 = \frac{x}{x+1}$$

$$\Rightarrow x^2 - 2x - 2 = 0$$

$$\Rightarrow x = \frac{2 \pm \sqrt{4 + 8}}{2} \begin{cases} \nearrow 1 + \sqrt{3} \\ \searrow 1 - \sqrt{3} \end{cases}$$

But $x > 0 \Rightarrow x = 1 + \sqrt{3}$

Ex: Find $[3; 5, \overline{2, 1}]$ explicitely

Ans: $x = [3; 5, s]$

$$= 3 + \cfrac{1}{5 + \cfrac{1}{s}}$$

$$= 3 + \frac{1 + \sqrt{3}}{5(1+\sqrt{3}) + 1} = 3 + \frac{1+\sqrt{3}}{6 + 5\sqrt{3}}$$

$$= \frac{126 - \sqrt{3}}{39}$$

Ex Prove that: $\sqrt{n^2+1} = [n; \overline{2n}]$

Pf

If $s = [\overline{2n}] = 2n + \frac{1}{[\overline{2n}]} = 2n + \frac{1}{s}$

then $[n, \overline{2n}] = n + \frac{1}{[\overline{2n}]}$

$$= n + \frac{1}{n + \sqrt{n^2+1}} = \sqrt{n^2+1}$$

when $s = n + \sqrt{n^2+1}$.

Exercise: 1) Find $\sqrt{11}$ in continued fraction expansion.

2) Calculate first three convergents

3) Using continued fraction expansion algorithm show that $\sqrt{n^2+1} = [n; \overline{2n}]$

4) Recall the algorithm for finding continued fraction expansion. We have numbers $\rho_1, \rho_2, \rho_3, \ldots$.

Show that, if $r = [a; a_1, a_2, \ldots]$ then

a) $r = [a; a_1, a_2, \ldots, a_{n-1}, \rho_n]$

b) $r_{2j} < r < r_{2j+1}, \forall j$

where r_n denotes the n'th convergent $[a; a_1, \ldots, a_n]$