Lecture 1

Recall, Finite Continued Fraction

1) $r = [a; a_1, \ldots, a_N] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 \cdots \frac{1}{a_N}}}$

2) Convergents $r_n := [a; a_1, \ldots, a_n]$

3) $r_n = \dfrac{s_n}{t_n}$, $\quad s_{n+1} = a_n s_n + s_{n-1}$
$\quad t_{n+1} = a_n t_n + t_{n-1}$

$$r_0 < r_2 < r_4 < \cdots \quad r_{..} < r_5 < r_3 < r_1$$

Today, infinite continued fraction

Theorem: Let $a \in \mathbb{Z}$ & $a_1, a_2, \ldots, \in \mathbb{N}$
Let $r_n := [a; a_1, \ldots a_n]$. Then the sequence r_n converges to a limit.
i.e. $\exists\, r \in \mathbb{R}$ s.t. $\forall\, \varepsilon > 0\ \exists\, N \geq 1$ s.t. $|r_n - r| < \varepsilon \quad \forall n > N$.

Pf (Non-examinable)

As $r_0 < r_2 < r_4 \cdots$ $\{r_{2j}\}$ form an increasing sequence. But they are bounded. So $r_{2j} \to \rho$. Similarly $r_{2j+1} \to \rho'$.

As $r_{2j} < r_{2j+1}$ we have $\rho \leq \rho'$. On the other hand,

$$|r_N - r_{N+1}| < \frac{1}{t_{N-1}\, t_N} \to 0.$$

as $N \to \infty$. Thus

$$|\rho - \rho'| \leq |\rho - r_N| + |r_N - r_{N+1}| + |r_{N+1} - \rho'|$$

If $N = 2j$, all of the above are small $\Rightarrow |\rho - \rho'| \to 0 \Rightarrow \rho = \rho'$!

Definition: We define the limit of the convergents to be the value of the infinite continued fraction $[a_0; a_1, a_2, \ldots]$

RMK: Every real number has a continued fraction expansion. We already know this for rational numbers.

Prop: For every irrational number $\exists\, a \in \mathbb{Z}$ & $a_1, a_2, \ldots \in \mathbb{N}$ s.t.

$$r = [a; a_1, a_2, \ldots]$$

Proof: We follow the algorithm for the rational numbers.

$$\lfloor r \rfloor = a, \quad \rho_1 = \frac{1}{r-a} > 1$$

$$\lfloor \rho_1 \rfloor = a_1, \quad \rho_2 = \frac{1}{\rho_1 - a_1} > 1$$

$$\left.\begin{array}{l} \rho_3, \rho_4 \ldots \ldots \\ a_2, a_3, \ldots \ldots \end{array}\right\} \text{we obtain.}$$

This process does not end. If it did r would have been a rational number (convince yourself!)

Thus $[a; a_1, a_2, \ldots]$ defines an infinite continued fraction. This is called the continued fraction algorithm. It remains to show that

$$[a; a_1 \ldots a_n] =: r_n \to r$$

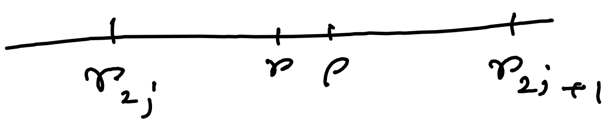
Exercise 1) Show that $[a; a_1, \ldots, a_{n-1}, \rho_n] = r$

Hint: Use induction

Exercise 2) Show that $r \geq r_{2j}$ & $r \leq r_{2j+1}$

Thus $|r - \rho| < |r_{2j} - r_{2j+1}| \to 0$

$\Rightarrow r = \rho$



$\square$

Rmk: This is exactly how we find continued fraction expansion of any real number.

<u>Prop</u> : Let $\gamma < \gamma'$ are positive real numbers. Then

$$[a; a_1, \dots, a_{n-1}, \gamma] < [a; a_1, \dots, a_{n-1}, \gamma']$$

if n is even

$$[a; a_1, \dots, a_{n-1}, \gamma] > [a; a_1, \dots, a_{n-1}, \gamma']$$

if n is odd.

<u>Proof</u> : $n = 1$

$$[a; \gamma] = a + \frac{1}{\gamma} > a + \frac{1}{\gamma'} = [a; \gamma']$$

$n = 2$

$$[a; a_1, \gamma] = a + \frac{1}{[a_1; \gamma]} < a + \frac{1}{[a_1; \gamma']}$$

$$= [a; a_1, \gamma']$$

.... Use induction for general n.

<u>Example</u> : Find continued fraction expansion of π.

<u>Ans</u> : Use the algorithm.

$$\lfloor \pi \rfloor = 3 \rightsquigarrow a = 3$$

$$\rho_1 = \frac{1}{\pi - 3} = 7.06\ldots \rightsquigarrow a_1 = 7$$

$$\rho_2 = \frac{1}{0.06\ldots} = 15.99\ldots \rightsquigarrow a_2 = 15$$

$$\rho_3 = \frac{1}{0.99\ldots} = 1.003\ldots \rightsquigarrow a_3 = 1$$

$$\rho_4 = \frac{1}{0.003\ldots} = 292.63\ldots \rightsquigarrow a_4 = 292$$

$\ldots$

$$\pi = [3; 7, 15, 1, 292, \ldots]$$

The convergents are $3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}$ etc.

<u>Ex</u>: $x = 1 + \frac{1}{2}\sqrt{2}$

$a = \lfloor x \rfloor = 1, \quad \rho_0 = 1$

$$\rho_1 = \frac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2} \rightsquigarrow a_1 = 1$$

$$\rho_2 = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 \rightsquigarrow a_2 = 2$$

$$\rho_3 = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 \rightsquigarrow a_3 = 2$$

$\ldots \qquad a_i = 2$

So, $r = [1; 1, 2, 2, \ldots, \ldots]$

<u>Ex</u>: $r = \sqrt{15} - 3$

$a = \lfloor r \rfloor = 0$ as $0 \leq \sqrt{15} - 3 < 1$
$(9 < 15 < 16)$

$r_1 = \dfrac{1}{\sqrt{15} - 3} = \dfrac{\sqrt{15} + 3}{6}$

$a_1 = \left\lfloor \dfrac{\sqrt{15} + 3}{6} \right\rfloor = 1$ as

$1 \leq \dfrac{\sqrt{15} + 3}{6} < 2 \iff 0 \leq \sqrt{15} - 3 < 6$

$r_2 = \dfrac{1}{\dfrac{\sqrt{15}+3}{6} - 1} = \dfrac{6}{\sqrt{15} - 3} = \sqrt{15} + 3$

$a_2 = \lfloor \sqrt{15} + 3 \rfloor = 6$ as

$6 \leq \sqrt{15} + 3 < 7 \iff 3 \leq \sqrt{15} < 4$
$\iff 9 \leq 15 < 16$

$r_3 = \dfrac{1}{\sqrt{15} - 3} = r_1$

so $r = [0; 1, 6, 1, 6, \ldots]$