Lecture 5B
MTH6102: Bayesian Statistical Methods

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Today's agenda

Today's lecture

- Review
- Compute posterior distribution for transformed parameters and multiple parameters
- Compute posterior estimates and credible intervals for transformed parameters and multiple parameters.
- Choose a prior distribution.

Review: point estimates

- Suppose we know the posterior distribution $p(\theta \mid y)$ for a one-dimensional parameter θ computed from

Posterior distribution $\propto$ prior distribution $\times$ likelihood

- We can obtain point estimates of θ by summarising the center of the posterior $p(\theta \mid y)$ using e.g.,
  - mean
  - median
  - mode
- We can also obtain a $1-\alpha$-probability or credible interval for θ.

Review: Credible or probability intervals

- A $1-\alpha$-probability or credible interval for θ is an interval (θ_L, θ_U) such that

$$P(\theta_L < \theta < \theta_U) = 1 - \alpha.$$

- The probabilities are calculated from the posterior distribution pmf or pdf

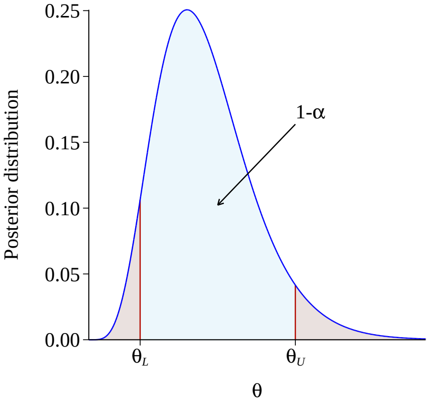
$$p(\theta \mid y)$$

Review: Equal tail intervals or symmetric probability intervals

- A (1-α)%equal-tail or symmetric probability interval is an interval (θ_L, θ_U) such that

$$P(\theta < \theta_L) = \alpha/2$$
$$P(\theta > \theta_U) = \alpha/2$$

- It's symmetric because the amount of probability remaining on either side of the interval is the same, $\alpha/2$.

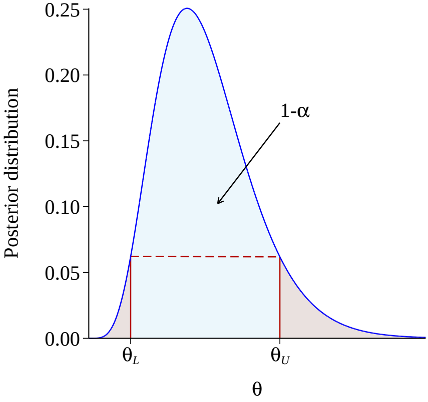


Review: Highest posterior density (HPD) intervals

- Let $p(\theta|y)$ be a unimodal posterior density for θ.
- A (1-α)% highest posterior density (HPD) interval is an interval (θ_L, θ_U) such that

$$P(\theta_L < \theta < \theta_U) = 1 - \alpha$$
$$p(\theta_L \mid y) = p(\theta_U \mid y)$$

- The interval captures the "most likely" values of the unknown.



Review: Highest posterior density (HPD) intervals

- Of all possible $(1-\alpha)\%$ credible intervals, the HPD interval is the **shortest.**
- If the density posterior density is unimodal and symmetric then the symmetric interval and the HPD interval coincide. Otherwise they do not.
- Finding a HPD interval in a non-symmetric distribution is not straightforward.

Transformed parameters

- Suppose we have arrived at a posterior distribution $p(\theta \mid y)$ for a parameter θ.
- Let $\psi = g(\theta)$, where g is a monotonic transformation of θ (increasing or decreasing), e.g., $\psi = \log(\theta)$, $\sqrt{\theta}$ or θ^3.
- **Questions:**
  - How do we make inferences about ψ?
  - Which posterior summary statements about θ carry over to ψ?
    -E.g. if $\tilde{\theta}$ is the posterior mean for θ, is $g(\tilde{\theta})$ the posterior mean for ψ?

Transforming random variables

- The shape of a probability density changes under nonlinear monotonic transformations of the random variable.
- Let g be a monotonic function.
- Suppose we have random variables X and Y with $Y = g(X)$.
- Their pdfs are related by

$$f_X(x) = |g'(x)|\, f_Y(g(x)) \quad \text{or}$$
$$f_Y(y) = \left|\frac{d}{dy} g^{-1}(y)\right| f_X(g^{-1}(y))$$

Example: Posterior of transformed parameters

- Bent coin with probability of success θ
- Flat prior on θ: $p(\theta) = 1$ for all $\theta \in [0, 1]$.
- $k = 5$ heads in $n = 6$ tosses.
- Find the posterior distribution of θ
- Find the posterior distribution of $\psi = \theta^3$.

Mean of transformed parameters

- Mean is NOT preserved by the transformation since for a nonlinear g

$$E(g(X)) \neq g(E(X)).$$

- So, if $\hat{\theta}_B$ is the posterior mean of θ, $g(\hat{\theta}_B)$ is NOT the posterior mean of ψ.

- The posterior density changes shape, so the mode is not preserved by the transformation.

- Also the endpoints of the highest posterior density credible intervals are not preserved.

Quantiles of transformed parameters

- Quantiles are preserved under nonlinear monotone transformations, so median is preserved.
- If θ_m be the posterior median for θ, then $g(\theta_m)$ is the posterior median for ψ.
- Similarly, equal tail credible intervals are preserved under increasing, one-to-one transformations
- If $(q_{0.025}, q_{0.975})$ is a 0.95 equal-tail credible intervals for θ, then $(g(q_{0.025}), g(q_{0.975}))$ is a 0.95 equal-tail credible intervals for $\psi = g(\psi)$ for any g monotonic increasing transformations.

More than one parameter

- We have covered one-parameter examples so far.
- We have considered conjugate priors - the simplest examples have one unknown parameter.
- Computational methods allow models with many parameters.
- And priors don't need to be conjugate.

Multiple parameters

- Let $\theta = (\theta_1, \ldots, \theta_K)$ be a vector of parameters.
- Then we can still use Bayes' theorem to compute the joint posterior

$$p(\theta_1, \ldots, \theta_K \mid y) \propto p(\theta_1, \ldots, \theta_K)\, p(y \mid \theta_1, \ldots, \theta_K)$$

- We still base our estimates on the joint posterior $p(\theta \mid y)$.
- For predictions of future data, we use the entire joint distribution.

Marginal distribution

- For point estimates of individual parameters, we typically use the marginal distribution.
- For example, if $\theta = (\theta_1, \theta_2, \theta_3)$, the marginal posterior distribution for θ_1 is
$$p(\theta_1 \mid y) = \int \int p(\theta_1, \theta_2, \theta_3 \mid y) \, d\theta_2 \, d\theta_3$$
- The computational methods used for Bayesian inference make going from joint to marginal distribution easy.
- No need to explicitly evaluate the integral.

Example: comparing two Binomials

- In a clinical trial, suppose we have n_1 control patients and n_2 treatment patients.
- x_1 control patients survive and x_2 treatment patients survive.
- Then, x_1 and x_2 are independent,

$$x_1 \sim \text{binomial}(n_1, p_1) \quad x_2 \sim \text{binomial}(n_2, p_2)$$

- We want to estimate $\tau = P(p_2 < p_1)$, the survival success for treatment group is lower than the survival success for control group.
- We might want to estimate the difference in proportions $d = p_2 - p_2$ or the log-odds $\log(\frac{p_1}{1-p_1})$

Example: comparing two Binomials

- The prior is $f(p_1, p_2) = 1, \quad 0 < p_1 < 1,\, 0 < p_2 < 1$
- By independence of the data, the posterior is

$$f(p_1, p_2 | x_1, x_2) \propto p_1^{x1}(1-p_1)^{n1-x1} p_2^{x2}(1-p_2)^{n2-x2}$$

- Notice that p_1, p_2 live on a square, and that

$$f(p_1, p_2 | x_1, x_2) \propto f(p_1|x_1) f(p_2|x_2),$$

  where $f(p_1|x_1) = p_1^{x1}(1-p_1)^{n1-x1}$, $f(p_2|x_2) = p_2^{x2}(1-p_2)^{n2-x2}$.
- Thus, p_1 and p_2 are independent under the posterior
- Also $f(p_1|x_1) \sim \text{beta}(x_1 + 1, n_1 - x_1 + 1)$,
  $f(p_2|x_2) \sim \text{beta}(x_2 + 1, n_2 - x_2 + 1)$

Simulation

- Let $P_{11}, \ldots, P_{1B}$ a random sample from beta$(x_1 + 1, n_1 - x_1 + 1)$.
- Let $P_{21}, \ldots, P_{2B}$ a random sample from beta$(x_2 + 1, n_2 - x_2 + 1)$.
- Then (P_{1i}, P_{2i}), $i = 1, \ldots, B$ is a sample from $f(p_1, p_2 | x_1, x_2)$.
- We estimate τ by counting the proportion of pairs (P_{1i}, P_{2i}) such that $P_{2i} < P_{1i}$