Solution to <u>PS 6 Q.7:</u>

Show that the energy of the solution $u(x,t)$ to the problem
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \geq 0, t \geq 0 \\ u(0,t) = 0 \\ u(x,0) = f(x), & u_t(x,0) = g(x) \end{cases} \quad (\star\star)$$
for the wave equation on <u>half line</u> is <u>conserved</u>.

Assuming f, g are compact support, use the conservation of energy to show uniqueness of solutions.

The Energy $E[u](t) = \int_0^\infty \left(\frac{1}{2} u_t^2 + \frac{1}{2} c^2 u_x^2\right) dx$

has time derivative

$$\frac{d}{dt} E[u](t) = \int_0^\infty \left(\frac{1}{2} \cdot 2 u_t u_{tt} + \frac{1}{2} c^2 \cdot 2 u_x u_{xt}\right) dx$$

$$= \int_0^\infty u_t \cdot u_{tt} \, dx + c^2 u_x \cdot u_t \Big|_0^\infty - \int_0^\infty c^2 u_{xx} \cdot u_t \, dx$$

(Integration by parts)

The boundary condition gives $u_t(0,t) = \frac{d}{dt} u(0,t) = 0$

The condition of compact support gives $u_x \cdot u_t$ is zero at ∞.

so $$\frac{d}{dt}E[u](t) = \int_0^\infty u_t \cdot (u_{tt} - c^2 u_{xx})\, dx$$

$$= \int_0^\infty u_t \cdot 0 \, dx$$

by the equation

$$= 0$$

so Energy is conserved.

Next, if u_1, and u_2 are 2 solutions to $(**)$

then $w = u_1 - u_2$ is a solution to

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \geq 0, t \geq 0. \\ u(0,t) = 0 \\ u(x,0) = 0, \quad u_t(x,0) = 0 \end{cases}$$

Noticing $E[w](0) = 0$, the preservation of energy implies $E[w](t) \equiv 0$ for all t.

so $w_t \equiv 0$, and thus

$$w(x,t) = \int_0^t w_t(x,t) + w(x,0) = 0 + 0 \equiv 0$$

Thus $u_1 - u_2 \equiv 0$ and $u_1 \equiv u_2$.

The solution is unique.