Solution to PS 5 Q6 :

Consider for $x \in [0, L]$ the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in [0, L] \\ u(0,t) = 0,\ u_x(L,t) = 0 & \leftarrow \text{Mixed boundary conditions} \\ u(x,0) = x,\ u_t(x,0) = 0 & \leftarrow \text{Initial conditions} \end{cases}$$

Find the solution by separation of variables.

Step 1: As we did in the lecture notes, we first consider solutions of the form $u(x,t) = X(x)\,T(t)$.

The equation becomes $X \cdot \ddot{T} - c^2 X'' T = 0$

(upper dot is t derivative and "prime" is x derivative)

So $$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X}$$ is independent of both x and t.

Thus $$\frac{\ddot{T}}{cT} = \frac{X''}{X} = -\lambda$$ is a constant.

This give 2 ODEs

$$\begin{cases} X'' + \lambda X = 0 & (a) \\ \ddot{T} + c^2 \lambda T = 0 & (b) \end{cases}$$

Using the boundary conditions

$u(0,t)=0, u_x(L,t)=0$, we get

$$X(0)=0, \quad X'(L)=0$$

Combining with (a), get an eigenvalue problem

$$\begin{cases} X''+\lambda X=0 \\ X(0)=0, \ X'(L)=0 \end{cases} \qquad (*)$$

Claim: $\lambda>0$.

proof of Claim: Multiply (a) by X and integrate, get

$$\int_0^L X\cdot X'' + \lambda\int X^2 = 0$$

$$X\cdot X'\Big|_0^L - \int (X')^2 + \lambda\int X^2 = 0$$

Using the boundary conditions, we have

$$X\cdot X'\Big|_0^L = X(L)X'(L) - X(0)X'(0) = 0$$

Since $X\neq 0$ is nontrivial, we have $\lambda>0$. #

Knowing $\lambda>0$, the general solution to $(*)$ is

$$X(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$$

The first boundary conditions read

$$0 = C_1\cdot\cos 0 + C_2\sin 0 = C_1$$

C_1, C_2 cannot be both zero because X is non-trivial.

so $C_2\neq 0$.

The second boundary condition is then

$$0 = X'(L) = C_2 \cdot \sqrt{\lambda}\cos(\sqrt{\lambda}x)\Big|_{x=L} = C_2\sqrt{\lambda}\cos(\sqrt{\lambda}L)$$

this implies $\sqrt{\lambda}L = \frac{\pi}{2} + n\pi$, $n = 1, 2, \cdots$

The eigenvalues are thus $\lambda_n = \frac{(\frac{1}{2}+n)^2\pi^2}{L^2}$

The eigenfunctions are $X_n(x) = \sin(\sqrt{\lambda_n}x)$

$$= \sin\frac{(\frac{1}{2}+n)\pi x}{L}$$

Knowing λ_n, we solve (b) and get

$$T_n(t) = a_n\cos\frac{(\frac{1}{2}+n)\pi ct}{L} + b_n\sin\frac{(\frac{1}{2}+n)\pi ct}{L}.$$

The general solutions are

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t)$$

$$= \sum_{n=1}^{\infty} a_n\sin\frac{(\frac{1}{2}+n)\pi x}{L}\cos\frac{(\frac{1}{2}+n)\pi ct}{L}$$

$$+ \sum_{n=1}^{\infty} b_n\sin\frac{(\frac{1}{2}+n)\pi x}{L}\sin\frac{(\frac{1}{2}+n)\pi ct}{L}$$

Step 2: Next, we use the initial values to determine the a_n's and b_n's.

Differentiate the general solution with respect to t, we get

$$u_t(x,t) = \sum_{n=1}^{\infty} -\frac{a_n\cdot(\frac{1}{2}+n)\pi c}{L}\sin\frac{(\frac{1}{2}+n)\pi x}{L}\sin\frac{(\frac{1}{2}+n)\pi ct}{L}$$

$$+ \sum_{n=1}^{\infty}\frac{b_n\cdot(\frac{1}{2}+n)\pi c}{L}\sin\frac{(\frac{1}{2}+n)\pi x}{L}\cos\frac{(\frac{1}{2}+n)\pi ct}{L}$$

The initial conditions then read (plugging in $t=0$)

$$\begin{cases} x = u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{(\frac{1}{2}+n)\pi x}{L} & \text{ⓒ} \\ 0 = u_t(x,0) = \sum_{n=1}^{\infty} b_n \cdot \frac{(\frac{1}{2}+n)\pi c}{L} \cdot \sin \frac{(\frac{1}{2}+n)\pi x}{L} & \text{ⓓ} \end{cases}$$

we get $b_n = 0$ for all n.

multiply the equation ⓒ by $\sin \frac{(\frac{1}{2}+m)\pi x}{L}$

and integrate from 0 to L, get

$$\int_0^L x \cdot \sin \frac{(\frac{1}{2}+m)\pi x}{L} = \sum_{n=1}^{\infty} a_n \int_0^L \sin \frac{(\frac{1}{2}+n)\pi x}{L} \sin \frac{(\frac{1}{2}+m)\pi x}{L} . \quad \text{ⓔ}$$

Recall $\int_0^L \sin \frac{(\frac{1}{2}+n)\pi x}{L} \sin \frac{(\frac{1}{2}+n)\pi x}{L} = \begin{cases} \frac{L}{2}, & n=m \\ 0, & n \neq m \end{cases}$

ⓔ then gives

$$a_m = \left[\int_0^L x \cdot \sin \frac{(\frac{1}{2}+m)\pi x}{L} \right] \cdot \frac{2}{L}$$

$$= \left[\frac{-x \cdot L \cdot \cos \frac{(\frac{1}{2}+m)\pi x}{L}}{(\frac{1}{2}+m)^2 \pi} \Bigg|_0^L + \int_0^L \frac{L \cdot \cos \frac{(\frac{1}{2}+m)\pi x}{L}}{(\frac{1}{2}+m)\pi} dx \right] \cdot \frac{2}{L}$$

$$= \left[\frac{-L^2 \cos(\frac{1}{2}+m)\pi}{(\frac{1}{2}+m)^2 \pi} + 0 + \frac{L^2 \sin \frac{(\frac{1}{2}+m)\pi x}{L}}{(\frac{1}{2}+m)^2 \pi^2} \Bigg|_0^L \right] \cdot \frac{2}{L}$$

$$= \left[0 + 0 + \frac{L^2 \sin\left(\frac{1}{2}+m\right)\pi}{\left(\frac{1}{2}+m\right)^2 \pi^2} - 0 \right] \cdot \frac{2}{L}$$

$$= \frac{8L \cdot (-1)^n}{(1+2m)^2 \pi}$$

And so

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8L \cdot (-1)^n}{\pi^2 \cdot (1+2n)^2} \sin\left[\left(\frac{1}{2}+n\right)\frac{\pi x}{L}\right] \cos\left[\left(\frac{1}{2}+n\right)\frac{\pi c t}{L}\right]$$