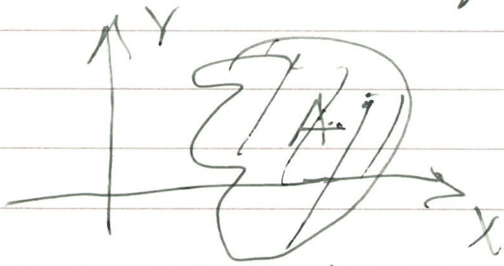
Transformation of two r.v.'s.

Statement of the problem.

X, Y are r.v.'s with pdf $f_{X,Y}(x,y)$.

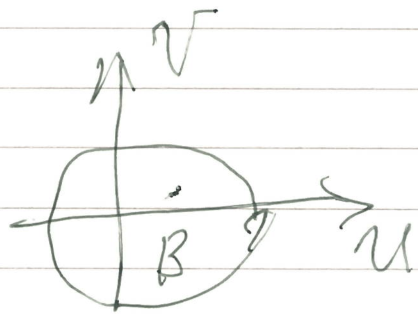
A is the domain of (X,Y) in $\mathbb{R}^2$.



$$\begin{cases} f_{X,Y}(x,y) > 0 \text{ in } A. \\ f_{X,Y}(x,y) = 0 \text{ otherwise} \end{cases}$$

We consider a transformation

$$\begin{cases} U = g_1(X,Y) \\ V = g_2(X,Y) \end{cases}$$



B is the image of A.

We suppose that there is a 1 to 1-correspondence between $(X,Y) \leftrightarrow (U,V)$

And hence, the inverse transformation exist and is

$$\begin{cases} X = h_1(U,V) \\ Y = h_2(U,V) \end{cases}$$

Question. What is $f_{U,V}(u,v)$?

<u>Answer.</u> Suppose that h_1, h_2 have continuous derivatives, and

$$J = \det \begin{vmatrix} \frac{\partial h_1(u,v)}{\partial u} & \frac{\partial h_1(u,v)}{\partial v} \\ \frac{\partial h_2(u,v)}{\partial u} & \frac{\partial h_2(u,v)}{\partial v} \end{vmatrix} \neq 0$$

for all $(u,v) \in B$.

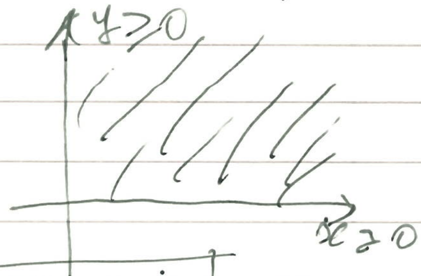
Then

$$f_{U,V}(u,v) = \begin{cases} f_{X,Y}(h_1(u,v), h_2(u,v)) \cdot |J| & \text{if } u,v \in B \\ 0 & \text{otherwise.} \end{cases}$$

<u>Example.</u> (X,Y) - r.v.'s with pdf

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{cases} U = X \\ V = X + Y \end{cases}$$



$$\begin{cases} X = U \\ Y = V - U \end{cases} \qquad \boxed{\begin{matrix} h_1(u,v) = u \\ h_2(u,v) = v - u \end{matrix}}$$

$$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1.$$

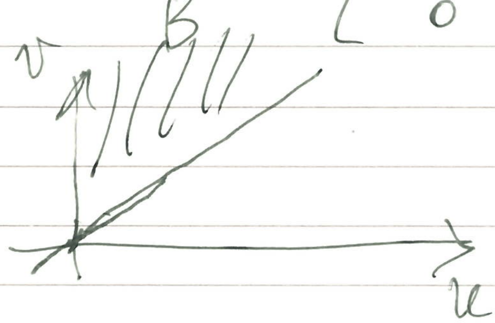
What is B? $\quad x \geq 0 \Leftrightarrow u \geq 0$

$y \geq 0 \Leftrightarrow v - u \geq 0.$

Hence, $B = \{V \geq U \geq 0\} = \{(u,v) : v \geq u \geq 0\}$ and

$$f_{U,V}(u,v) = \begin{cases} e^{-u-(v-u)} & \text{if } v \geq u \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-v} & \text{if } v \geq u \geq 0 \\ 0 & \text{otherwise} \end{cases}$$ ⊠

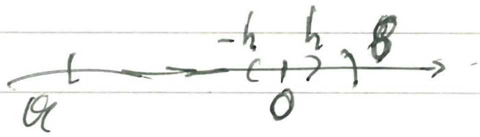


Moment generating functions.

Def-n. For a r.v. X, the moment generating function (mgf) in defined by

$$M_X(t) = \mathbb{E}(e^{tX}) \quad , -h < t < h$$

Remark. We consid $M_X(t)$ if it is defined in some interval (a, b) which contains 0



The $h > 0$ exists (this is required!).

If X is a continuous r.v. with pdf f_X, then

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

If X is a discret r.v. then

$$M_X(t) = \sum_x e^{tx} \mathbb{P}(X=x).$$

Properties of $M_X(t)$.

1. $M_X'(0) = \mathbb{E}(X)$
2. $M''(0) = \mathbb{E}(X^2)$
3. $Var(X) = M_X''(0) - (M_X'(0))^2$

"Proof": 1. From $M_X(t) = \mathbb{E}(e^{tX})$ we get

$$M_X'(t) = \frac{d}{dt} M_X(t) = \frac{d}{dt}\left(\mathbb{E}(e^{tX})\right) = \mathbb{E}\left(\frac{d}{dt} e^{tX}\right)$$

We suppose that the derivative can be moved insid the sign of expetation. Hence

$$M_X'(t) = \mathbb{E}(Xe^{tX}).$$

$$M_X'(0) = \mathbb{E}(X).$$

2. Similarly

$$M''_X(t) = \frac{d}{dt}\mathbb{E}(Xe^{tX}) = \mathbb{E}\left(\frac{d}{dt}(Xe^{tX})\right)$$

$$= \mathbb{E}(X^2 e^{tX}). \text{ Hence}$$

$$M''_X(0) = \mathbb{E}(X^2).$$

3. Follows from 1 and 2:

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = M''_X(0) - (M'_X(0))^2.$$

Remark. $M^{(k)}_X(0) = \mathbb{E}(X^k)$. $\boxtimes$

Example. $X \sim \mathrm{Exp}(\lambda)$. Find

(a) $M_X(t)$; (b) $\mathbb{E}(X)$, $\mathrm{Var}(X)$.

Solution. $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$

Hence

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_0^{\infty} e^{tx}\lambda e^{-\lambda x}\,dx$$

$$= \lambda\int_0^{\infty} e^{-(\lambda-t)x}\,dx \underset{\text{if } \lambda-t>0}{=} \frac{-\lambda}{\lambda-t} e^{-(\lambda-t)x}\Big|_{x=0}^{\infty}$$

$$= \frac{\lambda}{\lambda-t}, \text{ for } t < \lambda.$$

Answer to a). $X \sim \mathrm{Exp}(\lambda) \Rightarrow M_X(t) = \frac{\lambda}{\lambda-t}$, $t<\lambda$

(b) $\mathbb{E}(X) = \left(\frac{\lambda}{\lambda-t}\right)'_{t=0} = \frac{\lambda}{(\lambda-t)^2}\Big|_{t=0} = \frac{\lambda}{\lambda^2} = \lambda^{-1}$

$$\mathbb{E}(X^2) = \left(\frac{\lambda}{(\lambda-t)^2}\right)'\Big|_{t=0} = \frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0} = \frac{2\lambda}{\lambda^3} = 2\lambda^{-2}$$

Hence $Var(X) = 2\lambda^{-2} - (\lambda^{-1})^2 = \lambda^{-2}$

<u>Remark</u>. If $X \sim Exp(\lambda) \Rightarrow Var(X) = (\mathbb{E}(X))^2$.

<u>Example</u>. X has a Gamma distribution $Ga(\alpha, \beta)$. What is the mgf of X?

<u>Solution</u>. By the def-n of the Gamma distr-n,

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad \text{for } x > 0.$$

$$(f_X(x) = 0 \text{ if } x \le 0).$$

<u>Remark</u>

If $\alpha = 1$, $\beta = \lambda$, we get $Exp(\lambda)$.

So

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_0^{\infty} e^{tx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,dx$$

$$= \int_0^{\infty} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta-t)x}\,dx$$

We must have $t < \beta$. Then

$$M_X(t) = \frac{\beta^\alpha}{(\beta-t)^\alpha} \int_0^{\infty} \underbrace{\frac{(\beta-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta-t)x}}_{\text{pdf of } Ga(\alpha,\beta-t)}\,dx$$

Hence $\displaystyle\int_0^{\infty} \frac{(\beta-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta-t)x}\,dx = 1$

Finally, $M_X(t) = \frac{\beta^\alpha}{(\beta-t)^\alpha}$, $t<\beta$. ∎

<u>Method of moment generating functions</u>

<u>Theorem</u>. If X_1 and X_2 are random variables and $M_{X_1}(t) = M_{X_2}(t)$ then X_1 and X_2 have the same distribution.

<u>Example</u>. Suppose $Z \sim N(0,1)$, $Y = Z^2$.

(a) Find the distribution of Y using the mgf technique.

(b) Find $\mathbb{E}(Y)$, $Var(Y)$.

<u>Solution</u>. (a) We start by computing $M_Y(t)$.

$$M_Y(t) = \mathbb{E}(e^{tY}) = \mathbb{E}(e^{tZ^2})$$

$$= \int_{-\infty}^{\infty} e^{tz^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\frac{1}{2}-t)z^2} dz, \quad t < \frac{1}{2}.$$

We discussed the fact that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi}\,\sigma$$

Write $\left(\frac{1}{2}-t\right)z^2 = \frac{1}{2\sigma^2}z^2$

$$\sigma^2 = \frac{1}{1-2t}, \quad \sigma = \frac{1}{\sqrt{1-2t}}$$

Hence

$$\int_{-\infty}^{\infty} e^{-\left(\frac{1}{2}-t\right)z^2} dz = \sqrt{2\pi} \times \frac{1}{\sqrt{1-2t}}.$$

Finally

$$M_Y(t) = (1-2t)^{-\frac{1}{2}}$$

<u>**Answer**</u>. If $Y = Z^2$, $Z \sim N(0,1)$, then

$$M_Y(t) = (1-2t)^{-\frac{1}{2}}, \quad t < \frac{1}{2}.$$

So $Y \sim Ga\left(\frac{1}{2}, \frac{1}{2}\right)$.

(b) $\mathbb{E}(Y) = \frac{1}{2}(1-2t)^{-\frac{3}{2}} \times 2 \Big|_{t=0} = 1.$

$$\mathbb{E}(Y^2) = \frac{3}{2} \times 2 (1-2t)^{-\frac{5}{2}} \Big|_{t=0} = 3.$$

So $Var(Y) = 3 - 1^2 = 2$.

$\boxtimes$

<u>Corollary</u>. $Y \sim Ga\left(\frac{1}{2}, \frac{1}{2}\right)$. Hence

$$f_Y(y) = \frac{\sqrt{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} y^{-\frac{1}{2}} e^{-\frac{1}{2}y}$$

Summary for $Ga(\alpha, \beta)$

If $X \sim Ga(\alpha, \beta)$ which means, $\alpha > 0, \beta > 0$

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$$

$(f_X(x) = 0$ if $x \le 0)$.

Then

$$M_X(t) = \mathbb{E}(e^{tX}) = \frac{\beta^\alpha}{(\beta - t)^\alpha}, \quad t < \beta.$$

Application of mgf's: sums of independent random variables.

Theorem. If $X_1, X_2, \dots, X_n$ are independent r.v.'s with mgf's $M_{X_1}(t), M_{X_2}(t) \dots M_{X_n}(t)$, then the mgf of $Y = X_1 + X_2 + \dots + X_n$ is

$$M_Y(t) = M_{X_1}(t) \times M_{X_2}(t) \times \dots M_{X_n}(t).$$

$$\left(M_Y(t) = \prod_{i=1}^n M_{X_i}(t) \right).$$

Proof. We shall use the fact that

$$\mathbb{E}\left(\prod_{i=1}^n Z_i\right) = \prod_{i=1}^n \mathbb{E}(Z_i) \quad \text{if } Z_i \text{ are}$$

independent r.v.

By def-n, $M_Y(t) = \mathbb{E}(e^{tY})$

$$= \mathbb{E}\left(\exp\left(t \sum_{i=1}^n X_i\right)\right) = \mathbb{E}(\exp(tX_1 + tX_2 + \dots + tX_n))$$

$$= \mathbb{E}\left(\underbrace{e^{tX_1} \times e^{tX_2} \times \dots \times e^{tX_n}}_{\text{independent r.v.'s}}\right)$$

$$= E(e^{tX_1}) \times E(e^{tX_2}) \times \dots \times E(e^{tX_n})$$

$$= \prod_{i=1}^{n} M_{X_i}(t) \qquad \boxtimes$$

Example. $X_1, \dots, X_n$ are independent r.v.'s, $X_i \sim Exp(\lambda)$. What is the distribution of $Y = \sum_{i=1}^{n} X_i$.

Solution. We know $M_{X_i}(t) = \frac{\lambda}{\lambda - t}$, $t<\lambda$

Hence

$$M_Y(t) = \left(\frac{\lambda}{\lambda - t}\right)^n = \frac{\lambda^n}{(\lambda - t)^n}$$

so $Y \sim ga(n, \lambda) \Longrightarrow$

$$f_Y(y) = \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} \qquad \blacksquare$$

Example. Suppose $Y_1, Y_2, \dots, Y_n$ are independent r.v.'s, $Y_i \sim N(\mu_i, \sigma_i^2)$. Show that $Z = \sum_{i=1}^{n} Y_i \sim N\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$

Proof. $M_{X_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2 t}$ – we know that.

Then $M_Z(t) = \prod_{i=1}^{n} \exp(\mu_i t + \frac{1}{2}\sigma_i^2 t)$

$$= \exp\left(\left(\sum_{i=1}^{n} \mu_i\right) t + \frac{1}{2}\left(\sum_{i=1}^{n} \sigma_i^2\right) t\right)$$

We see that $M_Z(t)$ is the mgf of $N(\mu, \sigma^2)$, where

$$\mu = \sum_{i=1}^{n} \mu_i, \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2 \quad \boxtimes$$

Inequalities.

Theorem. (Markov's inequality). Suppose that X is a non-negative r.v. Then for any $\delta > 0$

$$\mathbb{P}(X \geq \delta) \leq \frac{\mathbb{E}(X)}{\delta}$$

Proof. Define a new r.v. Z by

$$Z = \begin{cases} 1 & \text{if } X \geq \delta \\ 0 & \text{if } X < \delta. \end{cases}$$

Note: $X \geq \delta Z$. Indeed, if $X \geq \delta$ then $Z = 1$ and $X \geq \delta \times 1 = \delta$. If $X < \delta$ then $Z = 0$ and $X \geq 0 = \underbrace{\delta \cdot 0}_{\delta Z}$

But then $\mathbb{E}(X) \geq \mathbb{E}(\delta Z) = \delta\, \mathbb{E}(Z)$

since $\mathbb{E}(Z) = 1 \times \mathbb{P}(Z=1) + 0 \times \mathbb{P}(Z=0)$

$= 1 \times \mathbb{P}(X \geq \delta) = \mathbb{P}(X \geq \delta)$

So $\mathbb{E}(X) \geq \delta\, \mathbb{P}(X \geq \delta)$ and

$$\mathbb{P}(X \geq \delta) \leq \frac{\mathbb{E}(X)}{\delta} \qquad \blacksquare$$

<u>Example</u>.

C.W.4 Tutorial, Week 5.

1) $f_{XY}(x,y) = \underbrace{c\,e^{-|x|}}_{h(x)} \times \underbrace{e^{-|y|}}_{g(y)}$ for all x, y.

$\Longrightarrow$ X, Y are independent.

2) X, Y - independent.

$g_1(X)$, $g_2(Y)$, - 2 new r.v.'s. Then

$$\mathbb{E}(g_1(X) \times g_2(Y)) = \mathbb{E}(g_1(X)) \times \mathbb{E}(g_2(Y)).$$

Proof. Suppose $f_{X,Y}(x,y)$ is the pdf of X, Y.

Then we know

$$\mathbb{E}(g_1(X)g_2(Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x) g_2(y) f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x) g_2(y) f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g_1(x) g_2(y) f_X(x) f_Y(y)\,dx \right) dy$$

$$= \int_{-\infty}^{\infty} \left(g_2(y) f_Y(y) \underbrace{\left[\int_{-\infty}^{\infty} g_1(x) f_X(x)\,dx \right]}_{\text{constant function of } y} \right) dy$$

$$= \int_{-\infty}^{\infty} g_1(x) f_X(x)\,dx \times \int_{-\infty}^{\infty} g_2(y) f_Y(y)\,dy$$

$$= \mathbb{E}(g(X)) \times \mathbb{E}(g(Y)).$$

(a) If $g_1(x) = x^k$, $g_2(y) = y^m$, we get

$$\mathbb{E}(X^k Y^m) = \mathbb{E}(X^k) \times \mathbb{E}(Y^m).$$

(b) $$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X) \times \mathrm{Var}(Y)}}$$

$$\mathrm{Cov}(X,Y) = \underbrace{\mathbb{E}(XY)}_{\mathbb{E}(X)\times\mathbb{E}(Y)} - \mathbb{E}(X)\times\mathbb{E}(Y) = 0.$$

<u>Exercise</u>. If $X_1, \dots, X_n$ are independent r.v.'s, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i).$$

3) $X \sim \mathrm{Uniform}(0,1)$. $f_X(x) = \begin{cases} 1, & 0<x<1 \\ 0 & \text{otherwise.} \end{cases}$

Y, $f_Y(y) = \begin{cases} 1 & \text{if } 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$

$$Z = X+Y \sim f_{X+Y}(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$

$0 < X+Y < 2$, $\Rightarrow z \in (0,2)$.

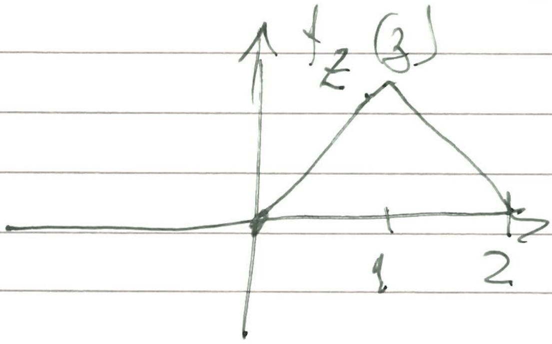
$$f_Z(z) = \int_0^1 f_X(z-y)\, dy = \int_0^z dy = z, \text{ if } z<1.$$

$0 < z-y < 1, 0<y<z<1+y,$

If $z > 1$, then $0 < y \leq 1$, but also $y > z-1$

$$\Rightarrow f_Z(z) = \int_{z-1}^{1} dy = y\Big|_{y=z-1}^{1} = 1-(z-1) = 2-z.$$

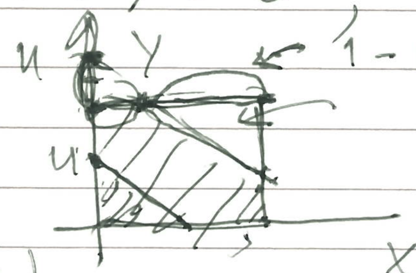
$$f_Z(z) = \begin{cases} z & z \in [0,1) \\ 2-z & z \in [1,2) \\ 0 & \text{otherwise.} \end{cases}$$



5) $U = X+Y$, $f_{X,Y}(x,y) = \begin{cases} 1 & x,y \in [0,1) \\ 0 & \text{otherwise} \end{cases}$

$F_U(u) = \mathbb{P}(U \leq u)$, $f_U(u) = F'(u)$.

$\mathbb{P}(U \leq 0) = 0$. $\mathbb{P}(U \leq 2) = 1$.



$\mathbb{P}(U < u)$

If $u \leq 1$, then $\mathbb{P}(U < u) = \frac{u^2}{2}$.

If $2 \geq u > 1$, $\mathbb{P}(U < u) = 1 - \frac{(2-u)^2}{2}$

$f_U(u) = u$ if $0 < u \leq 1$

$f_U(u) = 2-u$ if $1 < u \leq 2$,