MTH6107 Chaos & Fractals

Solutions 3

Exercise 1. Show that the notion of topological conjugacy defines an equivalence relation on the set of self-maps of $[-1, 1]$.

Recall that f and g are said to be topologically conjugate if there exists a homeomorphism $h : [-1, 1] \to [-1, 1]$ such that $h \circ f = g \circ h$

Clearly any f is topologically conjugate to itself: just take h to be the identity map.

The relation is symmetric: if $h \circ f = g \circ h$ then $H \circ g = f \circ H$ where $H = h^{-1}$.

The relation is transitive: if $h \circ f_1 = f_2 \circ h$ and $h' \circ f_2 = f_3 \circ h'$, then setting $H = h' \circ h$ we see that

$$H \circ f_1 = h' \circ h \circ f_1 = h' \circ f_2 \circ h = f_3 \circ h' \circ h = f_3 \circ H \, .$$

Therefore topological conjugacy is an equivalence relation.

Exercise 2. Use the map $h(x) = \sin(\pi x/2)$ to show that the map $f : [-1, 1] \to [-1, 1]$ defined by $f(x) = 1 - 2|x|$ is topologically conjugate to the Ulam map $g : [-1, 1] \to [-1, 1]$ given by $g(x) = 1 - 2x^2$.

First observe that $h : [-1, 1] \to [-1, 1]$ defined by $h(x) = \sin(\pi x/2)$ is indeed a homeomorphism.

We will show that $h \circ f = g \circ h$.

Firstly, if $x \in [-1, 0]$ then $h(f(x)) = \sin((2x + 1)\pi/2) = \sin(\pi/2 + \pi x) = \cos(\pi x)$, and if $x \in [0, 1]$ then $h(f(x)) = \sin((1 - 2x)\pi/2) = \sin(\pi/2 - \pi x) = \cos(\pi x)$.

Secondly, $g(h(x)) = 1 - 2\sin^2(\pi x/2) = \cos \pi x$.

So $g(h(x)) = h(f(x))$, as required.

Exercise 3. Determine whether the map $F : [-1, 1] \to [-1, 1]$ given by $F(x) = 1 - |x|$ is topologically conjugate to the map $G : [-1, 1] \to [-1, 1]$ given by $G(x) = 1 - x^2$.

The two maps are *not* topologically conjugate.

Justification: Every point in $[0, 1]$ has prime period 2 under F, whereas G only has a single orbit of prime period 2 (namely $\{0, 1\}$), therefore the maps cannot be topologically conjugate.

Henceforth let $D : [0, 1) \to [0, 1)$ be the doubling map $D(x) = 2x \pmod 1$, in other words

$$D(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2) \\ 2x - 1 & \text{for } x \in [1/2, 1) \, . \end{cases}$$

Exercise 4. For the map D, determine all its periodic points of period ≤ 5.

In general a point x has period n for the doubling map if and only if $x = j/(2^n - 1)$ for some integer j with $0 \le j \le 2^n - 2$.

The only fixed point is at 0.

The points of prime period 2 are $1/3$ and $2/3$.

The points of prime period 3 are $1/7$, $2/7$, $3/7$, $4/7$, $5/7$, $6/7$.

The points of prime period 4 are those points of the form $j/15$ which are not a fixed point or of prime period 2; in other words, $1/15$, $2/15$, $1/5$, $4/15$, $2/5$, $7/15$, $8/15$, $3/5$, $11/15$, $4/5$, $13/15$, $14/15$.

The points of prime period 5 are those points of the form $j/31$ for integers j with $1 \le j \le 30$.

Exercise 5. Write down the binary digit expansions for all the periodic points of D of period ≤ 5.

If x is periodic with $x = \sum_{k=1}^{\infty} b_k/2^k$, where each $b_k \in \{0,1\}$, then the binary digit sequence $(b_k)_{k=1}^{\infty}$ is periodic (see Exercise 9), so it suffices to give the corresponding periodic word.

The fixed point 0 corresponds to periodic word 0.

The period-2 point $1/3$ corresponds to periodic word 01 (i.e. $1/3 = .010101\ldots$), and the period-2 point $2/3$ corresponds to periodic word 10 (i.e. $2/3 = .101010\ldots$).

The period-3 point $1/7$ corresponds to periodic word 001 (i.e. $1/7 = .001001001\ldots$), the period-3 point $2/7$ corresponds to periodic word 010 (i.e. $2/7 = .010010010\ldots$), and the period-3 point $4/7$ corresponds to periodic word 100 (i.e. $4/7 = .100100100\ldots$).

The period-4 points $1/5, 2/5, 3/5, 4/5$ correspond, respectively, to periodic words 0011, 0110, 1001, 1100.

The period-4 points $1/15, 2/15, 4/15, 8/15$ correspond, respectively, to periodic words 0001, 0010, 0100, 1000.

The period-4 points $7/15, 11/15, 13/15, 14/15$ correspond, respectively, to periodic words 0111, 1011, 1101, 1110.

The period-5 points $j/31$ $(1 \le j \le 30)$ correspond to the 30 length-5 words on the alphabet $\{0,1\}$ which contain at least one 0 and at least one 1. For example $1/31$ corresponds to 00001 (i.e. $1/31 = .000010000100001\ldots$), etc.

Exercise 6. Determine the period-5 orbit of D which is contained in the interval $[3/20, 13/20]$.

The unique such orbit is $\{5/31, 10/31, 20/31, 9/31, 18/31\}$.

Exercise 7. Determine the periodic orbit of D which is contained in the interval $[3/10, 4/5]$.

The unique such orbit is $\{1/3, 2/3\}$.

Exercise 8. For all prime numbers $3 \le p \le 19$, determine the period (under the map D) of the point $1/p$.

$1/3$ has prime period 2.
$1/5$ has prime period 4.
$1/7$ has prime period 3.
$1/11$ has prime period 10.

$1/13$ has prime period 12.
$1/17$ has prime period 8.
$1/19$ has prime period 18.

Exercise 9. Given $x \in [0, 1)$, with binary expansion $x = \sum_{k=1}^{\infty} b_k/2^k$ where each $b_k \in \{0, 1\}$, show that x is periodic under D if and only if the binary digit sequence $(b_k)_{k=1}^{\infty}$ is periodic.

Applying the doubling map D corresponds to a (left) shift of the binary digit sequence, so if

$$x = .b_1 b_2 \dots b_T b_1 b_2 \dots b_T \dots$$

is such that the digit sequence has period T, then $D^T(x) = x$, so x is periodic under D.

Conversely, if x is periodic with period T, then $x = D^T(x) = 2^T x \pmod 1$, so $x(2^T - 1) =: m \in \{1, 2, \dots, 2^T - 2\}$, therefore

$$x = \frac{m}{2^T - 1} = \frac{m}{2^T} \frac{1}{1 - 2^{-T}} = \frac{m}{2^T} \left(1 + 2^{-T} + 2^{-2T} + 2^{-3T} + \dots\right) .$$

Now let $b_1, \dots, b_T \in \{0, 1\}$ be such that

$$m = b_1 2^{T-1} + b_2 2^{T-2} + \dots + b_T 2^0$$

so

$$\frac{m}{2^T} = \frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{b_T}{2^T} ,$$

therefore

$$x = \left(\frac{b_1}{2} + \frac{b_2}{2^2} + \dots + \frac{b_T}{2^T} \right) \left(1 + 2^{-T} + 2^{-2T} + 2^{-3T} + \dots\right) ,$$

in other words

$$x = .b_1 b_2 \dots b_T b_1 b_2 \dots b_T \dots ,$$

so the digit sequence is periodic.

Let $T : [0, 1) \to [0, 1)$ be the tripling map $T(x) = 3x \pmod 1$, in other words

$$T(x) = \begin{cases} 3x & \text{for } x \in [0, 1/3) \\ 3x - 1 & \text{for } x \in [1/3, 2/3) \\ 3x - 2 & \text{for } x \in [2/3, 1) . \end{cases}$$

Exercise 10. Determine whether or not D and T are topologically conjugate.

D and T are *not* topologically conjugate. To see this, note for example that D has a single fixed point in $[0, 1)$ (namely at 0), whereas T has two fixed points in $[0, 1)$ (namely 0 and $1/2$), and that if D and T were topologically conjugate then they would have had the same number of fixed points.

Exercise 11. Identify, with justification, those points of prime period 4 for D which are also of prime period 4 for T.

For D there are 3 orbits of prime period 4, namely $\{1/15, 2/15, 4/15, 8/15\}$, $\{1/5, 2/5, 4/5, 3/5\}$, and $\{7/15, 14/15, 13/15, 11/15\}$.

Under T, the orbit $\{1/5, 2/5, 4/5, 3/5\}$ has prime period 4, because $T(1/5) = 3/5$, $T(3/5) = 4/5$, $T(4/5) = 2/5$, $T(2/5) = 1/5$. Under T the points in $\{1/15, 2/15, 4/15, 8/15\}$ or $\{7/15, 14/15, 13/15, 11/15\}$ are pre-periodic but not periodic.

So the points with prime period 4 under both D and T are precisely $1/5, 2/5, 4/5, 3/5$.