MTH6107 Chaos & Fractals

Exercises 3

Exercise 1. Show that the notion of topological conjugacy defines an equivalence relation on the set of self-maps of $[-1,1]$.

Exercise 2. Use the map $h(x) = \sin(\pi x/2)$ to show that the map $f : [-1,1] \to [-1,1]$ defined by $f(x) = 1 - 2|x|$ is topologically conjugate to the map $g : [-1,1] \to [-1,1]$ given by $g(x) = 1 - 2x^2$.

Exercise 3. Determine whether the map $F : [-1,1] \to [-1,1]$ given by $F(x) = 1 - |x|$ is topologically conjugate to the map $G : [-1,1] \to [-1,1]$ given by $G(x) = 1 - x^2$.

Henceforth let $D : [0,1) \to [0,1)$ be the doubling map $D(x) = 2x \pmod 1$, in other words

$$D(x) = \begin{cases} 2x & \text{for } x \in [0,1/2) \\ 2x-1 & \text{for } x \in [1/2,1)\,. \end{cases}$$

Exercise 4. For the map D, determine all its periodic points of period ≤ 5.

Exercise 5. Write down the binary digit expansions for all the periodic points of D of period ≤ 5.

Exercise 6. Determine the period-5 orbit of D which is contained in the interval $[3/20, 13/20]$.

Exercise 7. Determine the periodic orbit of D which is contained in the interval $[3/10, 4/5]$.

Exercise 8. For all prime numbers $3 \le p \le 19$, determine the period (under the map D) of the point $1/p$.

Exercise 9. Given $x \in [0,1)$, with binary expansion $x = \sum_{k=1}^{\infty} b_k/2^k$ where each $b_k \in \{0,1\}$, show that x is periodic under D if and only if the binary digit sequence $(b_k)_{k=1}^{\infty}$ is periodic.

Let $T : [0,1) \to [0,1)$ be the tripling map $T(x) = 3x \pmod 1$, in other words

$$T(x) = \begin{cases} 3x & \text{for } x \in [0,1/3) \\ 3x - 1 & \text{for } x \in [1/3,2/3) \\ 3x - 2 & \text{for } x \in [2/3,1)\,. \end{cases}$$

Exercise 10. Determine whether or not D and T are topologically conjugate.

Exercise 11. Identify, with justification, those points of prime period 4 for D which are also of prime period 4 for T.