Let's investigate period-2 points in detail.

Period-2 pts are solutions to the equation

$$f_\mu^2(x) = x$$

i.e. $f_\mu^2(x) - x = 0$

i.e. $f_\mu(\mu x(1-x)) - x = 0$

i.e. $\mu . \mu x(1-x) . (1 - \mu x(1-x)) - x = 0$

i.e. $-x + (\mu^2 x - \mu^2 x^2)(1 - \mu x + \mu x^2) = 0$

Check this

i.e. $\underbrace{x(-\mu x + \mu - 1)}_{\text{Note this is } f_\mu(x) - x \text{ i.e. solutions are fixed pts}} \underbrace{(\mu^2 x^2 - (\mu^2 + \mu)x + (\mu+1))}_{\text{Least-period 2 pts will be roots of this quadratic}} = 0$

The points of least period 2 are the roots of

$$\mu^2 x^2 - (\mu^2 + \mu)x + \mu + 1$$

i.e. The two points, p_+ and p_-, of least period 2 are

$$p_\pm = \frac{1}{2\mu^2}\left(\mu^2+\mu \pm \sqrt{(\mu^2+\mu)^2 - 4\mu^2(\mu+1)}\right)$$

$$= \frac{1}{2\mu}\left(\mu+1 \pm \sqrt{(\mu+1)(\mu-3)}\right)$$

Notice that if $\mu = 3$ then both p_+ and p_- are equal to the fixed point $\frac{\mu-1}{\mu} = 2/3 = \frac{1}{2\mu}\left(\mu+1 \pm \sqrt{0}\right)$.

If $\mu < 3$ then $p_\pm$ do not exist (we do not consider non-real solutions).

Is the 2-cycle $\{p_+, p_-\}$ attracting?
Is it repelling?

More precisely, for which parameter values μ is the 2-cycle attracting/repelling?

We will analyse the modulus of the multiplier M of the 2-cycle, where

$$M = |(f_\mu^2)'(p_+)|$$
$$= |f_\mu'(f_\mu(p_+)) \cdot f_\mu'(p_+)|$$
$$= |f_\mu'(p_-) \cdot f_\mu'(p_+)|$$
$$= |(\mu - 2\mu p_-) \cdot (\mu - 2\mu p_+)|$$
$$= |\mu^2 - 2\mu^2(p_- + p_+) + 4\mu^2 p_- p_+|$$
$$= |\mu^2 - 2\mu(\mu+1) + 4(\mu+1)|$$
$$= |4 + 2\mu - \mu^2|$$

Recall that the 2-cycle is attracting if $M < 1$, and repelling if $M > 1$.

Now $M = 1$ when either ① $4 + 2\mu - \mu^2 = 1$

or ② $4 + 2\mu - \mu^2 = -1$

Case ①: $4 + 2\mu - \mu^2 = 1 \iff 0 = \mu^2 - 2\mu - 3$

$\iff 0 = (\mu - 3)(\mu + 1)$

So $M = 1$ when $\mu = 3$ (we ignore the case $\mu = -1$ since we only consider $\mu \in [0, 4]$)

Case ②: $4 + 2\mu - \mu^2 = -1$

$\iff \mu^2 - 2\mu - 5 = 0$

$\iff \mu = \frac{1}{2}\left(2 \pm \sqrt{4 + 20}\right)$

$= \frac{1}{2}\left(2 \pm 2\sqrt{6}\right)$

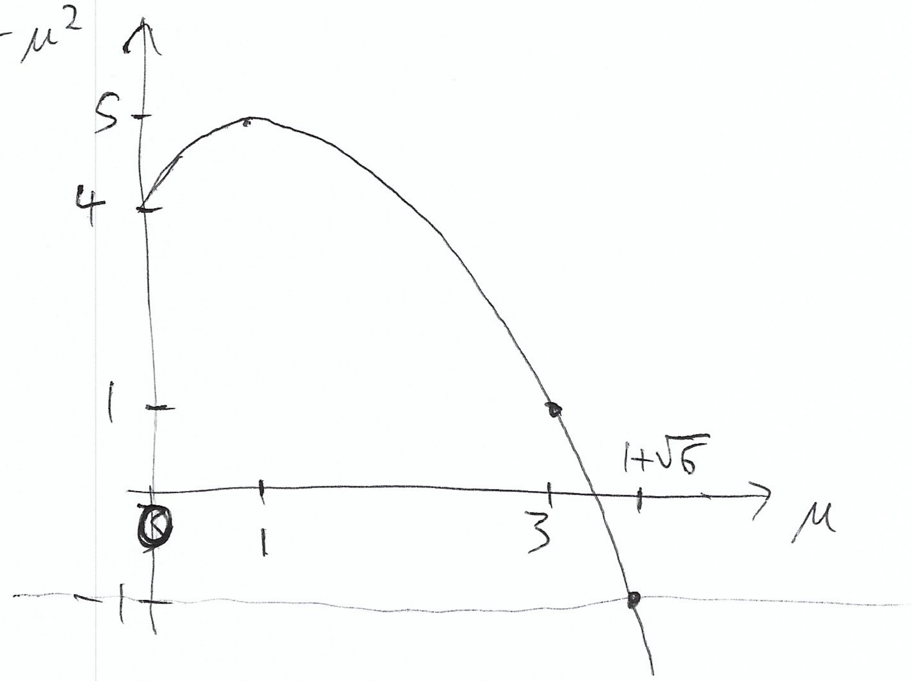
$= 1 \pm \sqrt{6}$

So $M = 1$ when $\mu = 1 + \sqrt{6} \simeq 3.449\ldots$

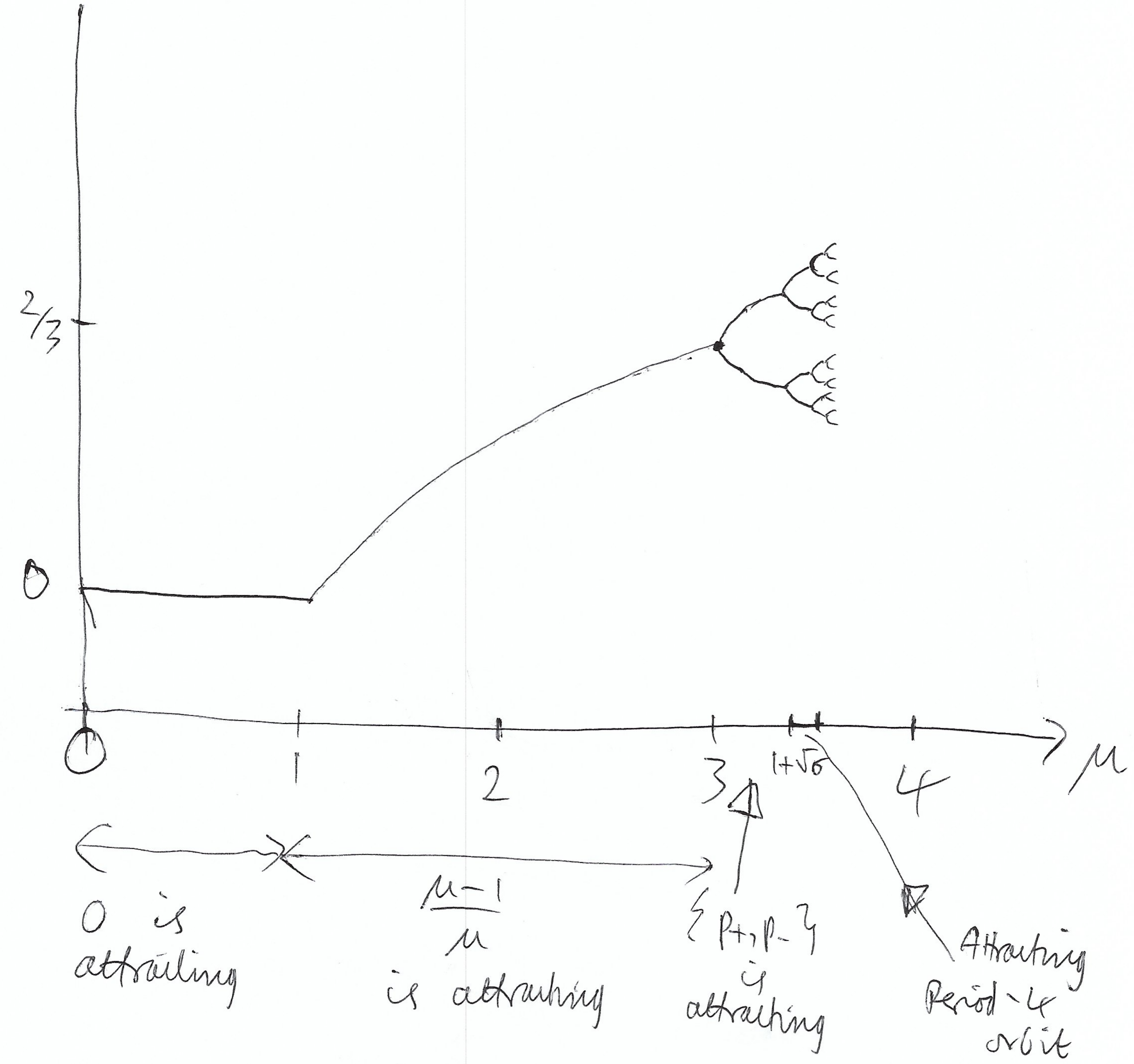
(we ignore the value $1 - \sqrt{6}$ since it is negative)

So $M < 1$ for $3 < \mu < 1+\sqrt{6}$

i.e. The 2-cycle $\{p_+, p_-\}$ is attracting if $3 < \mu < 1+\sqrt{6}$, and it is repelling if $\mu > 1+\sqrt{6}$



The evolution of the attracting periodic cycle, as μ increases, can be summarised by the following so-called <u>bifurcation diagram</u>:

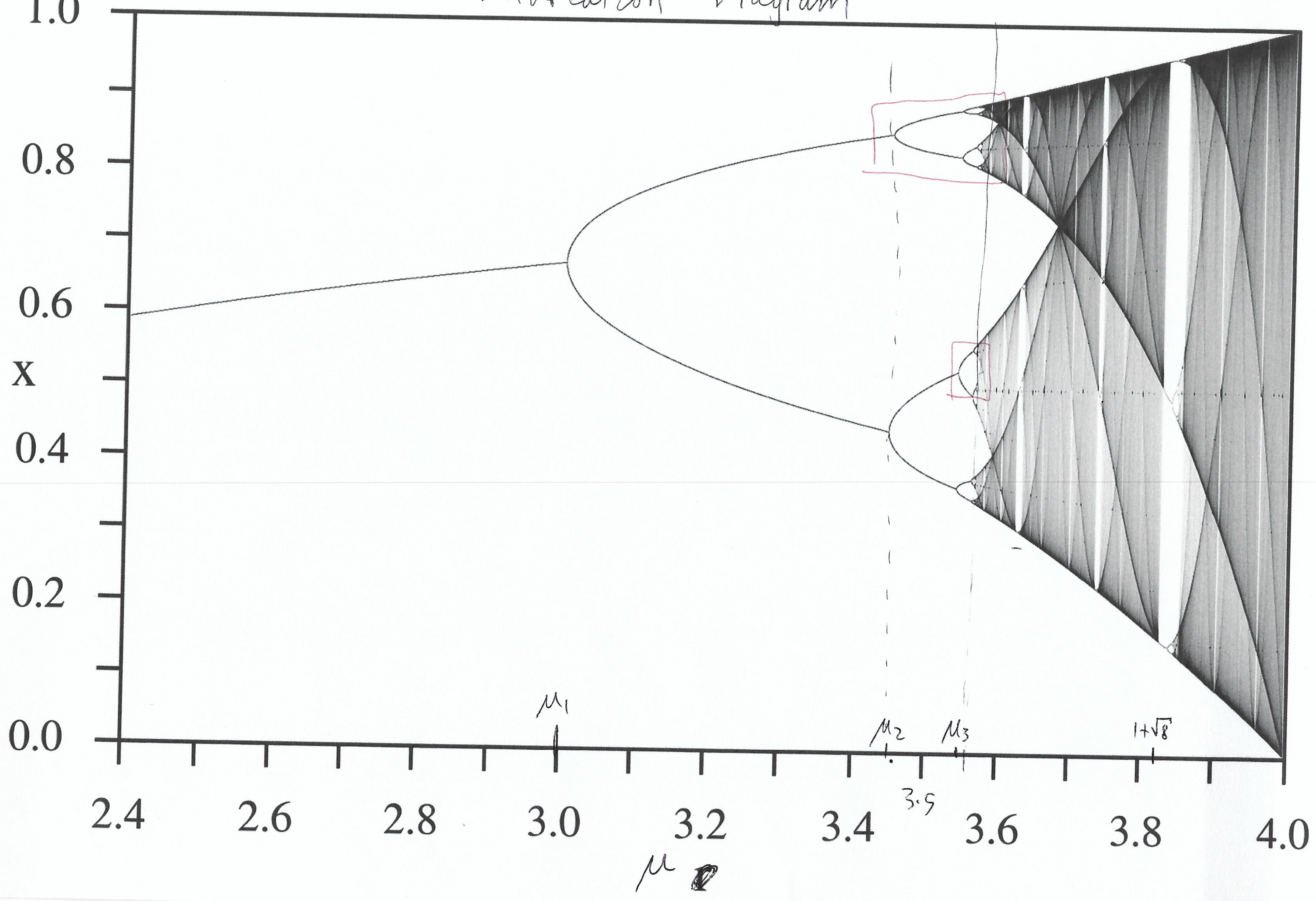


It turns out that at each stage, the loss of "attractingness"/attraction of the period-2^n orbit gives rise to an attracting periodic orbit of least period 2^{n+1}

This is called a <u>period-doubling bifurcation</u>.

In fact, since we have many such period-doubling bifurcations, this is sometimes referred to as a <u>period-doubling cascade</u>.

Bifurcation Diagram



If we let μ_n denote the special values of μ at which these period-doubling bifurcations occur then:

$$(\mu_0 = 1)$$

$$\mu_1 = 3$$

$$\mu_2 = 1 + \sqrt{6} \simeq 3.449$$

$$\mu_3 = 3.544\ldots$$

$$\mu_4 = 3.564\ldots$$

$$\mu_5 = 3.568\ldots$$

$$\vdots$$

$$\mu_\infty = 3.569946\ldots = \lim_{n\to\infty} \mu_n$$

The sequence $(\mu_n)_{n=0}^{\infty}$ converges to some value μ_∞, as $n \to \infty$.

On (μ_0, μ_1), $\frac{\mu-1}{\mu}$ is attracting

On (μ_1, μ_2), a 2-cycle is attracting

On (μ_2, μ_3), a 4-cycle is attracting

$\vdots$

On (μ_n, μ_{n+1}), a 2^n-cycle is attracting

The lengths of these bifurcation parameter intervals is decreasing rapidly (in fact, at some geometric rate, i.e. exponentially fast)

We can write

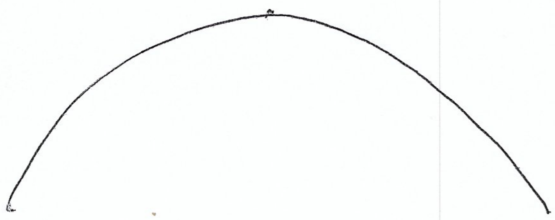
$$d_k = \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k}$$

It was observed experimentally by Feigenbaum (and others), around 1975, that the sequence (d_k) has a limit $d_\infty \simeq 4.669202\ldots$

This is called the *Feigenbaum constant* or the *Feigenbaum ratio*.

It was also observed that we get the *same* "universal value" d_∞ for *all* parametrised families of maps *looking like* the logistic family

"quadratic-like" maps



e.g. $g_\lambda(x) = \lambda \sin \pi x$

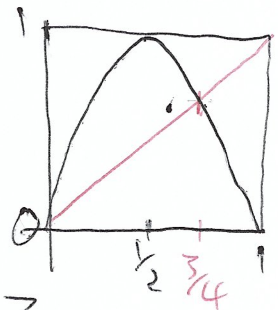
A natural question to pose is: what happens for $\mu > \mu_\infty$?

All period-2^n orbits are repelling beyond the value μ_∞.

For some parameters μ there are other attracting periodic orbits whose period is not a power of 2.

For some parameters μ, the behaviour of f_μ is "chaotic".

Example $\mu = 4$: $f(x) = f_4(x) = 4x(1-x)$



Fixed points: $f(x) = x$

$$\iff 4x - 4x^2 = x$$

$$\iff x = 0 \text{ or } 4x = 3$$

$$\iff x = 0 \text{ or } x = 3/4 \left(= \frac{\mu - 1}{\mu}\right)$$

Now $f'(x) = 4 - 8x$,

so $|f'(x)| = \begin{cases} 0 & \text{if } x = 0 \\ |-2| = 2 & \text{if } x = 3/4 \end{cases}$

so both fixed points are repelling.

Period-2 points: $f^2(x) = x$

$\iff f(f(x)) = x$

$\iff f(4x(1-x)) = x$

$\iff 4 . 4x(1-x)(1-4x(1-x)) = x$

$\iff -x(4x-3)(16x^2-20x+5) = 0$

So the roots of this quadratic

$p_\pm = \dfrac{5 \pm \sqrt{5}}{8}$ constitute a 2-cycle.

This 2-cycle is repelling, because:

$|(f^2)'(p_+)| = |f'(p_+) . f'(p_-)|$

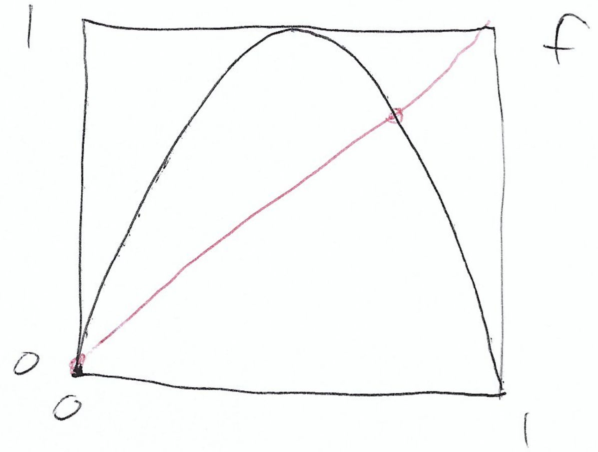
$= \ldots = |-4| = 4 > 1.$

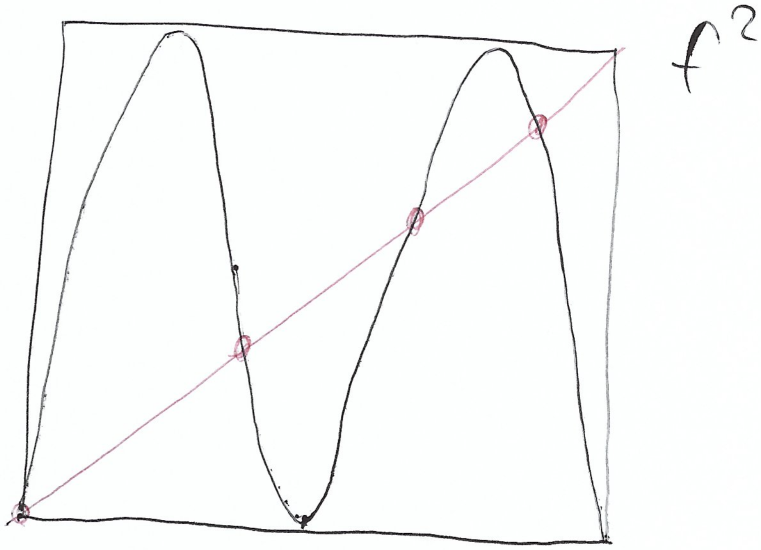
Period-3 points : $f^3(x) = x$

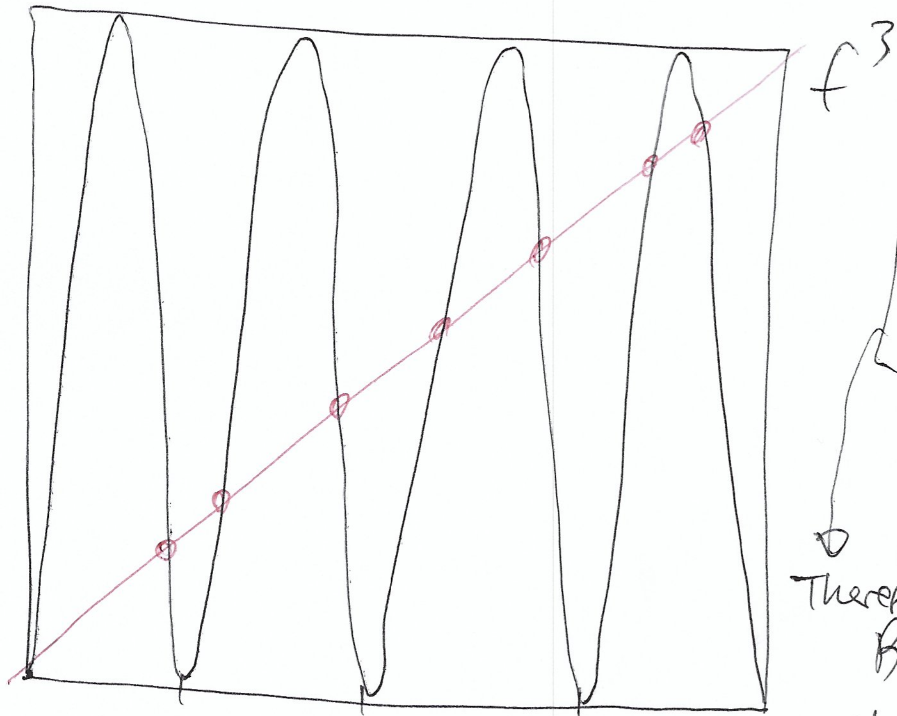
$f^3(x) - x$ ⊢ degree-8 polynomial

$\shortparallel$

$x(4x-3)$(cubic factor)(cubic factor)







There are two 3-cycles for f. Both 3-cycles are repelling.

Therefore By Sharkovskii's Theorem, this f has n-cycles for all $n \in \mathbb{N}$.

Topological conjugacy

Defn If X and Y are intervals in $\mathbb{R}$, we say $h : X \to Y$ is a homeomorphism if h is a bijection (i.e. invertible) and both h and h^{-1} are continuous.

Defn Let X and Y be intervals in $\mathbb{R}$. Assume $f : X \to X$, and $g : Y \to Y$. We say that $h : X \to Y$ is a topological conjugacy from f to g if

(i) h is a homeomorphism

(ii) $h \circ f = g \circ h$

Note regarding terminology:

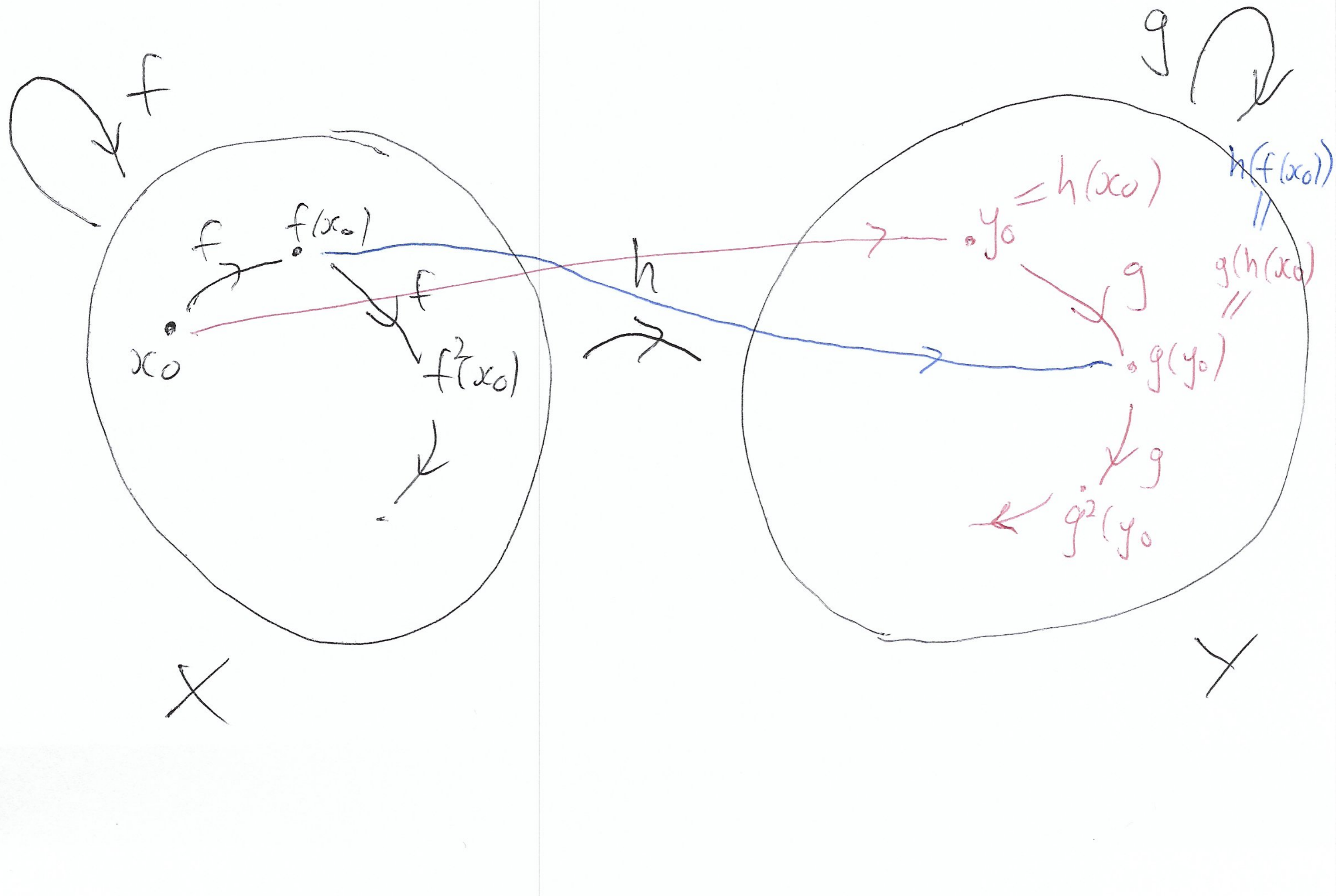
We also refer to ~~an~~ h as a topological conjugacy between f and g, or between g and f

We say that f and g are <u>topologically conjugate</u> if an h as above exists.

We also say that f is <u>topologically conjugate</u> to g.

<u>Remarks</u>

(a) Conjugacy can be viewed as a <u>change of coordinates / variables</u>.



So, if we want to study the orbit of x_0 under f in X, we can use h to equivalently study the orbit of $y_0 = h(x_0)$ under g in Y.

(b) The conjugacy equation $h \circ f = g \circ h$ can (of course!) be written as

$$f = h^{-1} \circ g \circ h$$

or $$g = h \circ f \circ h^{-1}$$

or $$f \circ h^{-1} = h^{-1} \circ g$$

i.e. This means $h(f(x)) = g(h(x))$ for all $x \in X$.

(c) Another way of thinking of the conjugacy equation (popular with algebraists) is that it is saying that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "h"] \\
Y \arrow[r, "g"'] & Y
\end{tikzcd}$$

Example We can find a topological conjugacy from the logistic map

$$f_\mu(x) = \mu x(1-x) \qquad (\mu > 0)$$

to $g_c(x) = x^2 + c$, for some suitable value of c.

To see this, we shall view f_μ and g_c as maps $\mathbb{R} \to \mathbb{R}$ (i.e. $X = Y = \mathbb{R}$).

Let's guess that the conjugacy h is of the form $h(x) = \alpha x + \beta$ $(\alpha \neq 0)$ (i.e. assume h is linear/affine).

Such an h is clearly a homeomorphism

We want to find α, β such that

$$h(f_\mu(x)) = g_c(h(x)) \quad \forall x \in \mathbb{R}$$

i.e. $\alpha f_\mu(x) + \beta = g_c(\alpha x + \beta)$

i.e. $\alpha \mu x(1-x) + \beta = (\alpha x + \beta)^2 + c$

i.e. $\alpha \mu x - \alpha \mu x^2 + \beta = \alpha^2 x^2 + 2\alpha\beta x + \beta^2 + c$

Comparing coefficients we get:

x^2 : $-\alpha\mu = \alpha^2$, i.e. $-\mu = \alpha$

x : $\alpha\mu = 2\alpha\beta$, i.e. $\mu = 2\beta$
i.e. $\beta = \mu/2$

constant term : $\beta = \beta^2 + c$ i.e. $c = \beta - \beta^2$
$= \mu/2 - (\mu/2)^2$

So, for $c = \mu/2 - (\mu/2)^2$ then f_μ and g_c are indeed topologically conjugate, where the topological conjugacy h is given by

$$h(x) = -\mu x + \mu/2$$

e.g. So for example if $\mu = 4$, i.e. we are considering $f_4(x) = 4x(1-x)$, then this is topologically conjugate to

$$g_{-2}(x) = x^2 - 2$$

because $c = \frac{\mu}{2} - \left(\frac{\mu}{2}\right)^2$

$$= \frac{4}{2} - \left(\frac{4}{2}\right)^2 = 2 - 4 = -2$$

[Recall that yesterday we looked at $f(x) = 4x(1-x)$, and ~~ca~~ several weeks ago we studied the map $x \mapsto x^2 - 2$]