Machine Learning with Python

MTH786U/P 2023/24

Lecture 7: Review for mid-term

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LINEAR ALGEBRA

Matrices and Vectors

Matrix: $$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1d} \\ x_{21} & x_{22} & \dots & x_{2d} \\ \vdots & & \ddots & \vdots \\ x_{s1} & x_{s2} & \dots & x_{sd} \end{pmatrix} \in \mathbb{R}^{s \times d}$$

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The row times column rule can be written as

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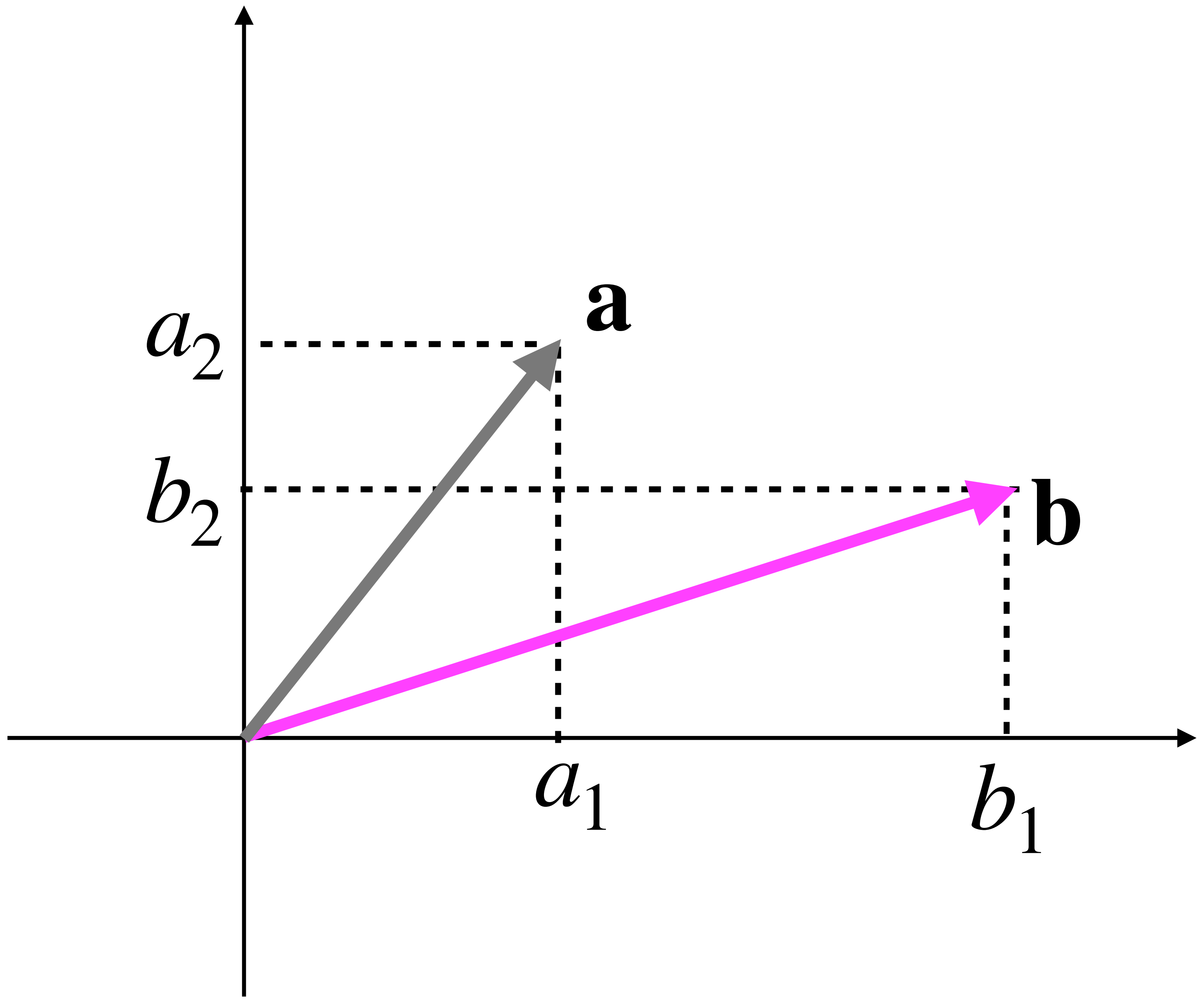
Inner/dot/scalar product



$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^{2\times 1}$$

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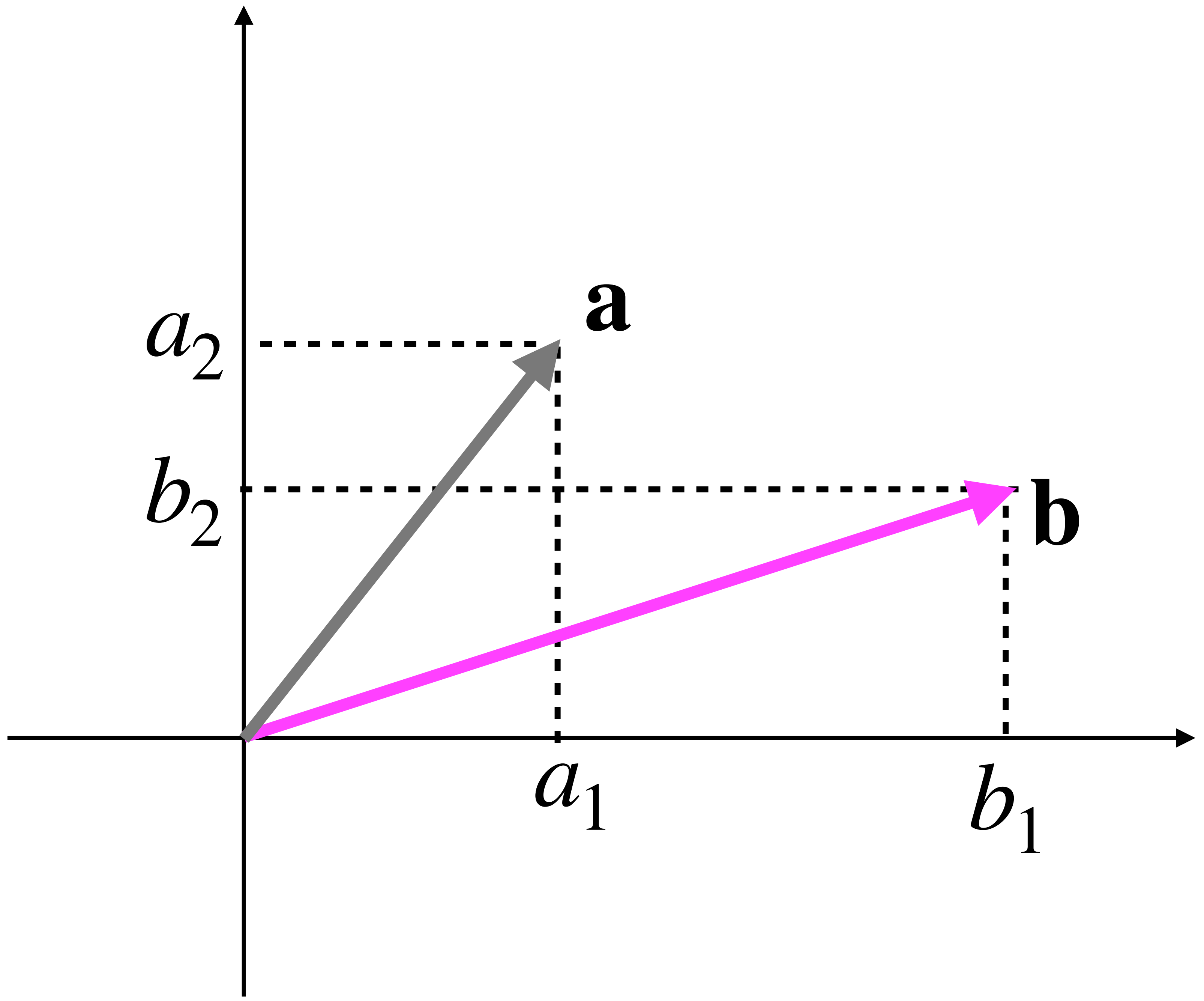

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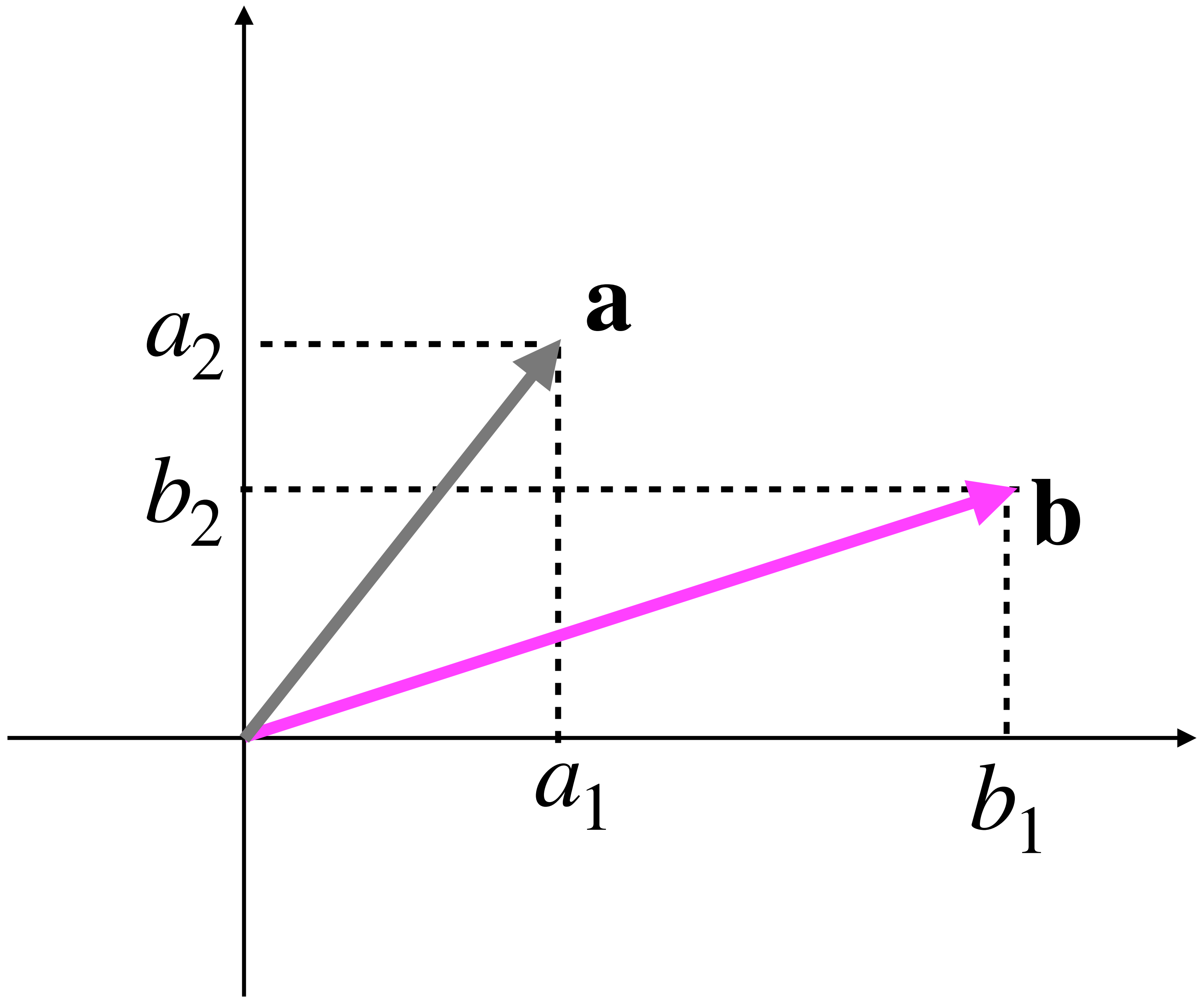

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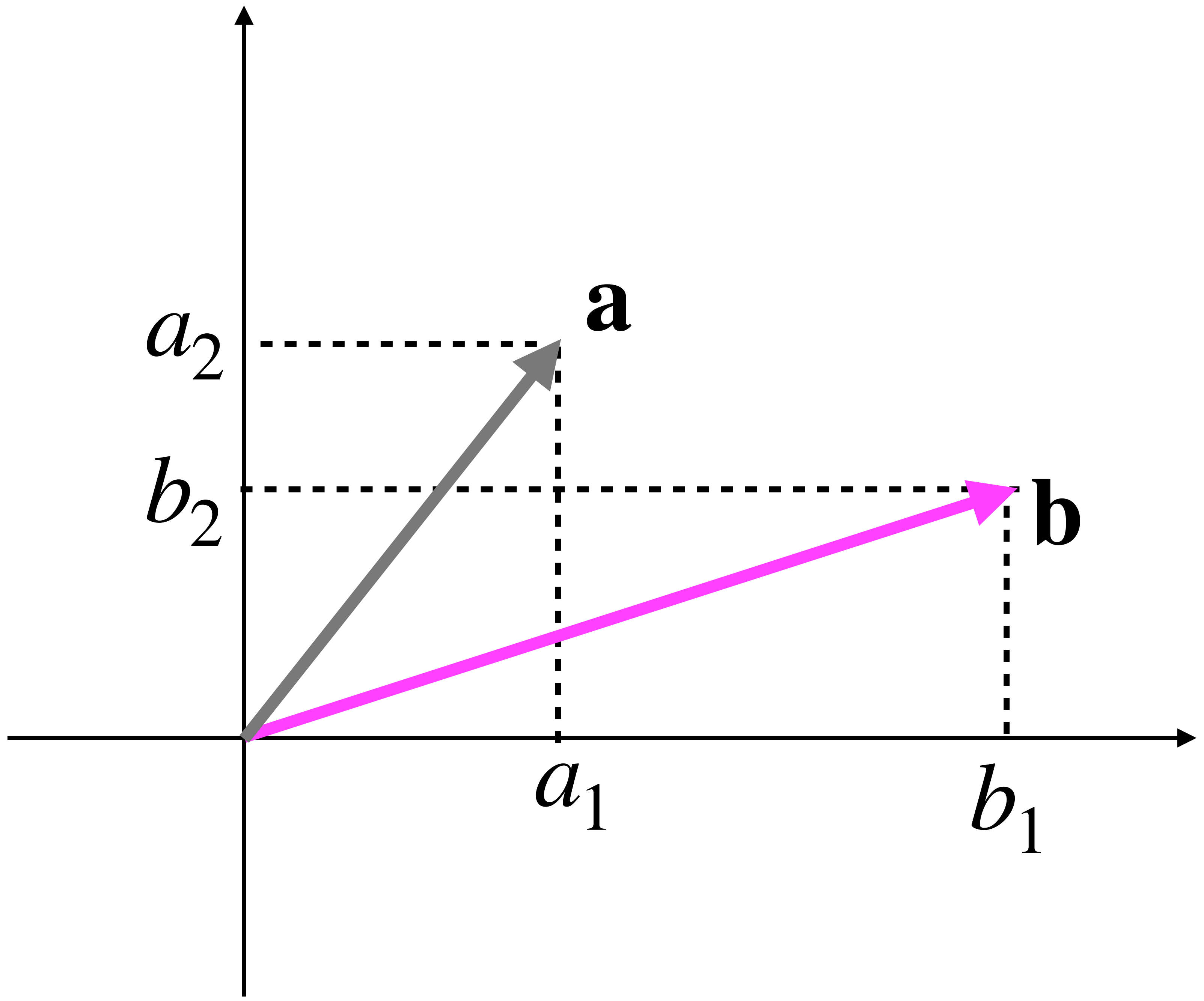

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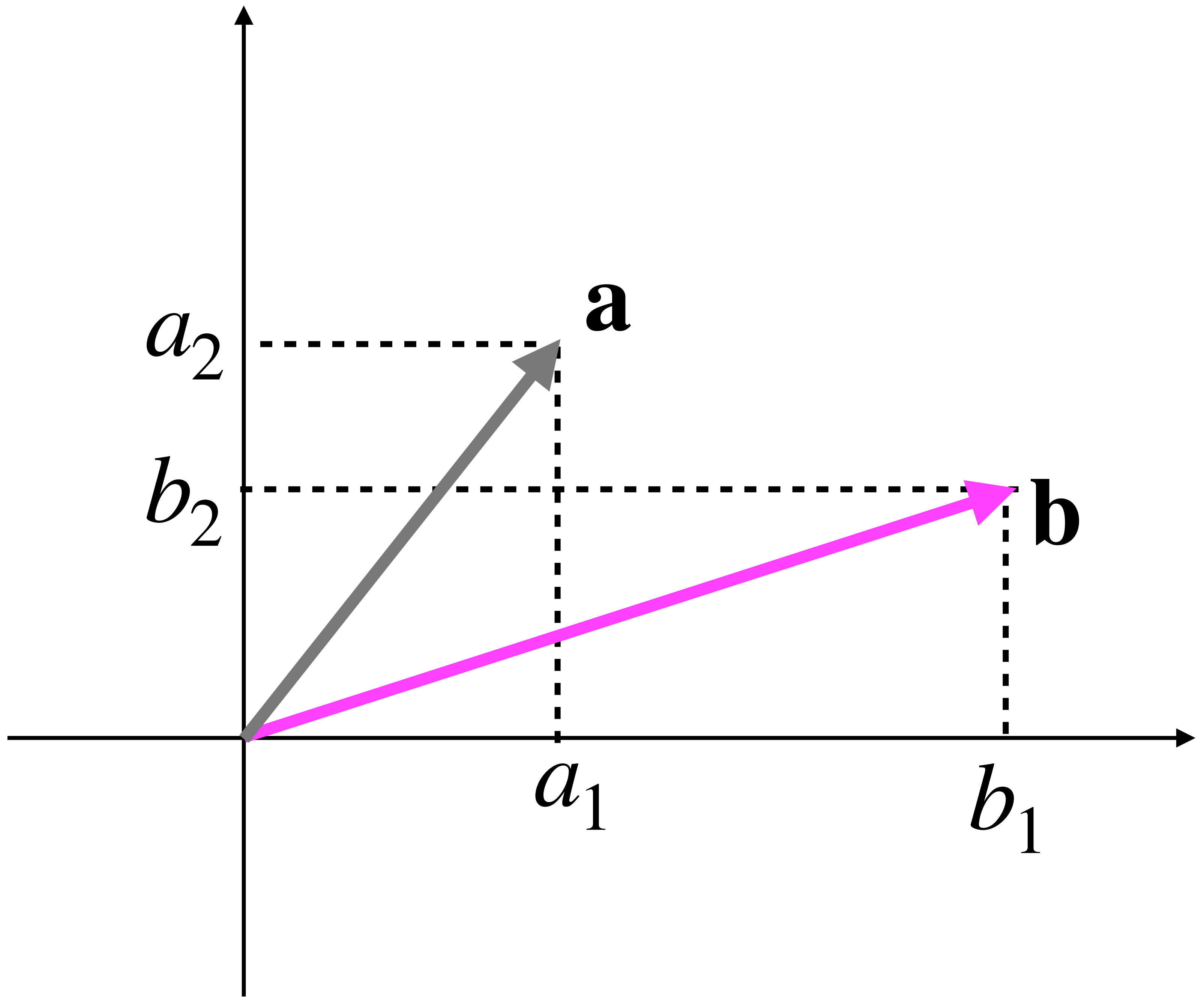

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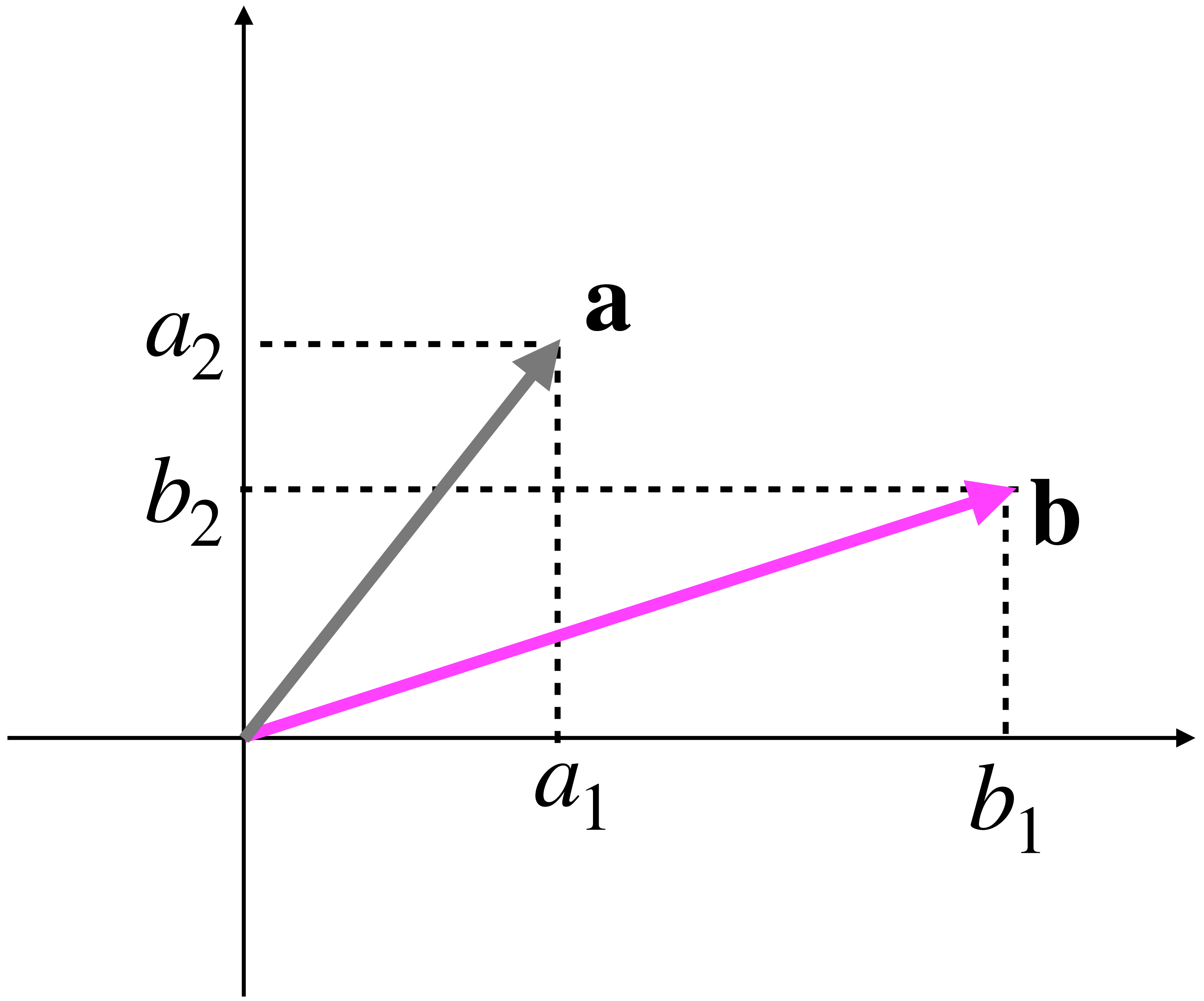

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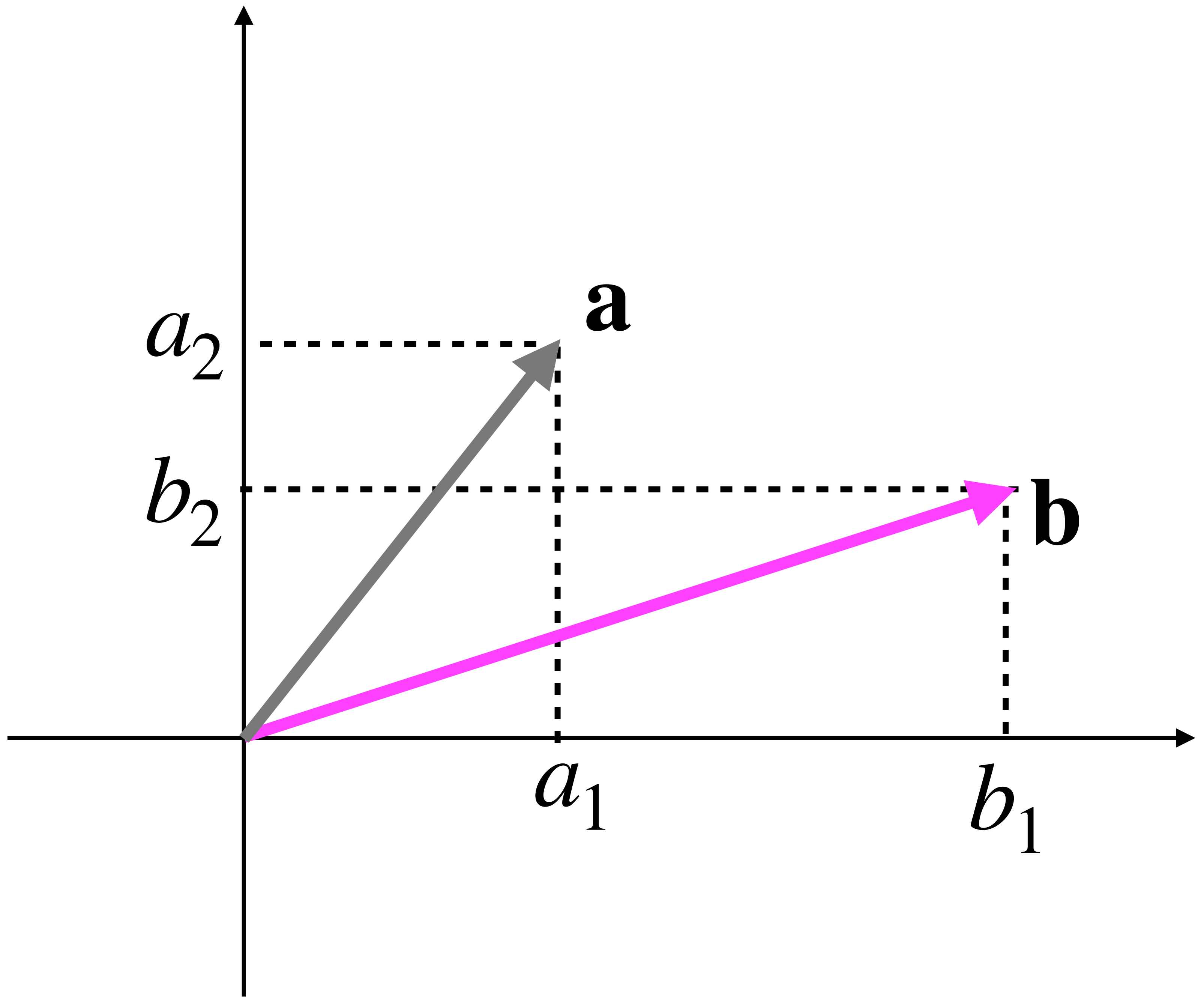

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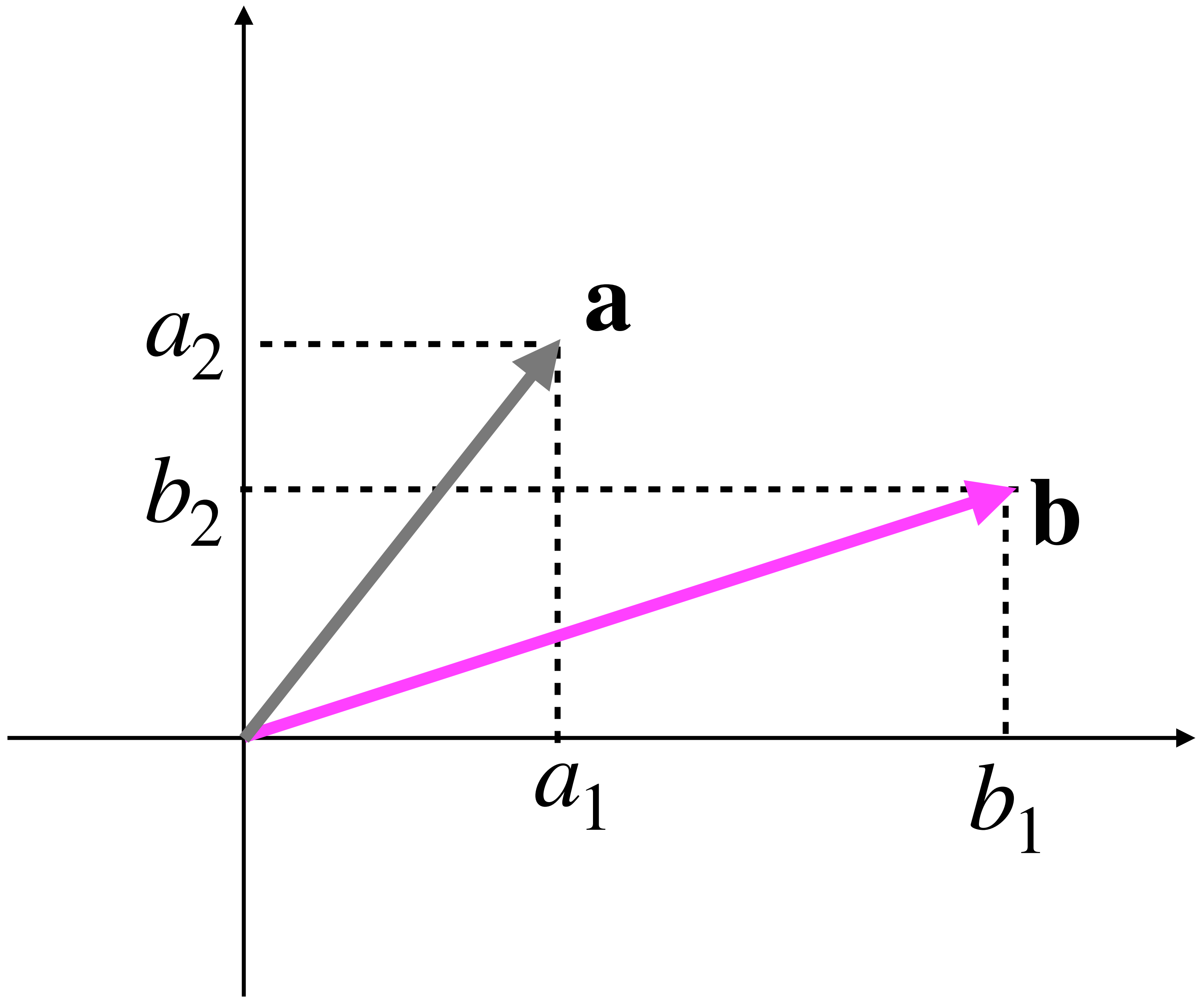

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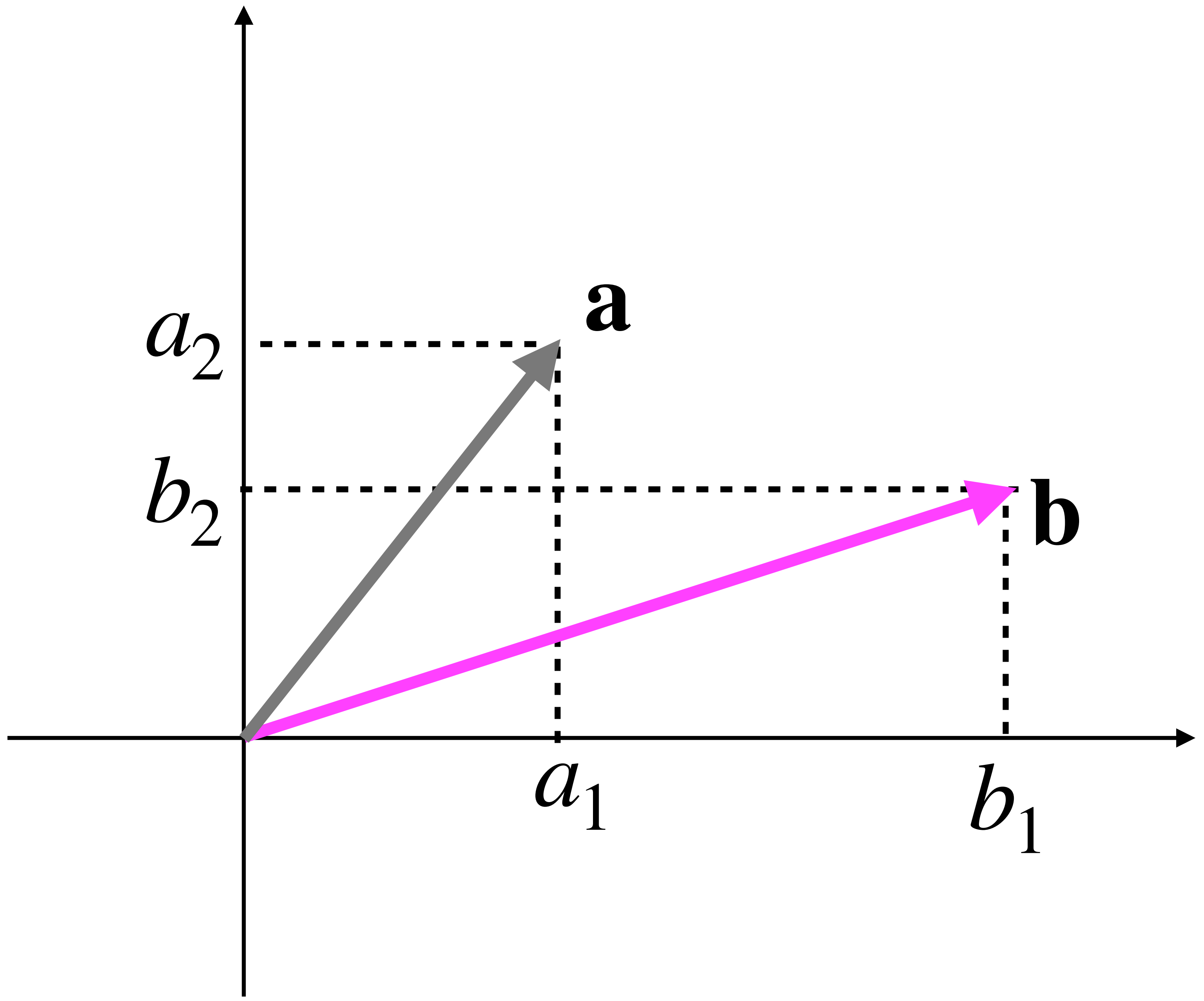

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Norm:

$$\|\mathbf{a}\| := \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} := \sqrt{\sum_{j=1}^{d} a_j^2}$$

Eigenvalues and Eigenvectors

Given a squared matrix $\mathbf{X} \in \mathbb{R}^{d \times d}$ a vector $\mathbf{w_i} \in \mathbb{R}^{d \times 1}$ that satisfies this equation

$$\mathbf{X}\mathbf{w_i} = \lambda_\mathbf{i}\mathbf{w_i}$$

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It is called eigenvector of the matrix and lambda is the correspondent eigenvalue

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Hence the goal is find solutions of this so-called characteristic equation

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Symmetric matrices lead to d real solutions and orthogonal eigenvectors!

(i) Find eigenvalues, eigenvectors and eigenvalue decomposition of matrix A.

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The eigenvalues of matrix A can be found by solving $\det(A - \lambda I) = 0$. In our case one has

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 4 \\ 0 & 5-\lambda \end{pmatrix} = (3-\lambda)(5-\lambda) = 0 \Rightarrow \lambda_{1,2} = 3, 5.$$

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Corresponding eigenvectors can be found by solving $Au = \lambda u$ for the values of λ found above.

$$Au^{(1)} = 3u^{(1)} \Rightarrow \begin{cases} 3u_1^{(1)} + & 4u_2^{(1)} = 3u_1^{(1)} \\ & 5u_2^{(1)} = 3u_2^{(1)} \end{cases} \Rightarrow u^{(1)} = (1, 0)^T .$$

$$Au^{(2)} = 5u^{(2)} \Rightarrow \begin{cases} 3u_1^{(2)} + & 4u_2^{(2)} = 5u_1^{(2)} \\ & 5u_2^{(2)} = 5u_2^{(2)} \end{cases} \Rightarrow u^{(2)} = (2, 1)^T .$$

$$\begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} u_1^{(1)} \\ u_1^{(2)} \end{pmatrix} = \lambda_1 \begin{pmatrix} u_1^{(1)} \\ u_1^{(2)} \end{pmatrix} = \begin{pmatrix} 3u_1^{(1)} \\ 3u_1^{(2)} \end{pmatrix}$$

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Given a $\mathbf{X} \in \mathbb{R}^{s \times d}$ the transpose is $\mathbf{X}^\top \in \mathbb{R}^{d \times s}$ hence $\mathbf{X}^\top \mathbf{X} \in \mathbb{R}^{d \times d}$

Singular value decomposition (SVD)

It can be shown that for any matrix $\mathbf{X} \in \mathbb{R}^{s \times d}$ there exist $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min(s,d)}$ vectors $\mathbf{U}_i \ \ (i \in [0,s])$ and vectors $\mathbf{V}_j \ \ (j \in [1,d])$ such that

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$$\mathbf{X}\mathbf{V}_i = \sigma_i \mathbf{U}$$

$$\mathbf{U}_i = \sigma_i^{-1} \mathbf{X}\mathbf{V}_i$$

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$$\mathbf{X}\mathbf{V}_i = \sigma_i \mathbf{U} \qquad \mathbf{U}_i = \sigma_i^{-1}\mathbf{X}\mathbf{V}_i \qquad \mathbf{V}_i = \sigma_i^{-1}\mathbf{X}^\top\mathbf{U}_i$$

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Important for rectangular matrices

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We then define $\mathbf{V} \in \mathbb{R}^{d \times d}$ such that of $\mathbf{X}^\top \mathbf{X} \mathbf{V_i} = \sigma_{\mathbf{i}}^{\mathbf{2}} \mathbf{V_i}$

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Singular value decomposition (SVD)

Summarizing

$$\mathbf{X}^\top \mathbf{X}\mathbf{V} = \sigma^2 \mathbf{V}$$

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Also

$$\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top$$

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$$\sigma_{1,2}^2 = \frac{7 \pm \sqrt{13}}{2}$$

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Notice the dimension of this vector!

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$$\mathbf{U}_2 = \sigma_2^{-1}\mathbf{B}\mathbf{V}_2$$

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$$= b\sqrt{\frac{2}{7-\sqrt{13}}}\begin{pmatrix}4-\sqrt{13}\\ \frac{\sqrt{13}-1}{2}\\ 0\end{pmatrix}$$

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Important note:

1) The matrix $\mathbf{B}\mathbf{B}^\top$ is a 3x3 matrix with another eigenvalue equal to zero
2) the associate vector needs to be orthogonal to the others, hence

$$\mathbf{U}_3 = (0,0,d)^\top$$

for data points $(x^{(1)}, y^{(1)})$ with $x^{(1)} = -c$ and $y^{(1)} = 2$, $(x^{(2)}, y^{(2)})$ with $x^{(2)} = 0$ and $y^{(2)} = 2$, and $(x^{(3)}, y^{(3)})$ with $x^{(3)} = c$ and $y^{(3)} = 2$, for some constant $c > 0$.

1. Derive the normal equation for this problem.

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3. Repeat the previous exercise, but this time assume you make an error in your measurement. The new, perturbed measurements $\mathbf{y}_\delta$ read $y_\delta^{(1)} = 2 + \varepsilon$, $y_\delta^{(2)} = 2 + \varepsilon$ and $y_\delta^{(3)} = 2 - \varepsilon$.

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$$= \frac{\varepsilon}{c}\sqrt{1 + \left(\frac{c}{3}\right)^2} > \frac{\varepsilon}{c}.$$

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The error in reconstruction is dominated by the ratio ε/c. If $c \ll \varepsilon$ the error can get potentially very large compared to the data error $\delta = \|y - y^\delta\| = \varepsilon\sqrt{3}$, which does not depend on c. Suppose $\varepsilon = 1/100$ and $c = 1/1000$, then $\delta \approx 0.01732$ but $\varepsilon/c = 10$. Hence, the data error δ is amplified by a factor larger than 577 in the reconstruction.

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$$\begin{cases} \sigma_1 = \sqrt{c^2 + 2 + \sqrt{c^4 + 4}}, \\ \sigma_2 = \sqrt{c^2 + 2 - \sqrt{c^4 + 4}} \end{cases}$$

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$$\begin{pmatrix} u_1^{(j)} \\ u_2^{(j)} \end{pmatrix} = \sigma_j^{-1} \begin{pmatrix} \dfrac{\sigma_j^2(1-c)+2c}{\sqrt{4+(\sigma_j^2-2)^2}} \\ \dfrac{\sigma_j^2(1+c)-2c}{\sqrt{4+(\sigma_j^2-2)^2}} \end{pmatrix}$$

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$$\hat{w} = \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

4. Repeat the previous exercise, but this time assume you make an error in your measurement. Consider two cases of the new, perturbed measurements

- $\mathbf{y}_\delta$ reads $y_\delta^{(1)} = 1 - \varepsilon$, $y_\delta^{(2)} = 1 + \varepsilon$.
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Repeating the previous exercise with the perturbed data $\mathbf{y}_\delta = \begin{pmatrix} 1-\varepsilon & 1+\varepsilon \end{pmatrix}^\top$ yields the normal equation

$$\begin{pmatrix} 2 & 2 \\ 2 & 2+2c^2 \end{pmatrix} \hat{\mathbf{w}}_\delta = \begin{pmatrix} 1 & 1 \\ 1-c & 1+c \end{pmatrix} \mathbf{y}_\delta$$
$$= \begin{pmatrix} 2 \\ 2+2c\varepsilon \end{pmatrix},$$

with the solution

$$\hat{\mathbf{w}}_\delta = \begin{pmatrix} 1 - \frac{\varepsilon}{c} \\ \frac{\varepsilon}{c} \end{pmatrix}.$$

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For the perturbed data $\mathbf{y}_\delta = \begin{pmatrix} 1+\varepsilon & 1+\varepsilon \end{pmatrix}^\top$ the normal equation takes the form

$$\begin{pmatrix} 2 & 2 \\ 2 & 2+2c^2 \end{pmatrix} \hat{\mathbf{w}}_\delta = \begin{pmatrix} 1 & 1 \\ 1-c & 1+c \end{pmatrix} \mathbf{y}_\delta$$
$$= \begin{pmatrix} 2+2\varepsilon \\ 2+2\varepsilon \end{pmatrix},$$

with the solution

$$\hat{\mathbf{w}}_\delta = \begin{pmatrix} 1+\varepsilon \\ 0 \end{pmatrix}.$$

5. In both cases compute the error between $\hat{\mathbf{w}}$ and $\hat{\mathbf{w}}_\delta$ in the Euclidean norm and compare with the data error $\delta := \|\mathbf{y} - \mathbf{y}_\delta\|$?

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CALCULUS

Calculus

Assume we have a function $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_d)$

Where $\mathbf{x} = (x_1, x_2, \ldots, x_d)$

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The gradient is defined as

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1} f(\mathbf{x}) \\ \frac{\partial}{\partial x_2} f(\mathbf{x}) \\ \vdots \\ \frac{\partial}{\partial x_d} f(\mathbf{x}) \end{pmatrix}$$

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It is a vector that indicates the direction of fastest increase of the function

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Then
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$$\frac{\partial}{\partial x_2} f(x) = 2x_1 (x_1 x_2 - y)$$

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$$\Rightarrow \qquad \nabla f(x) = 2 \begin{pmatrix} x_1 x_2^2 - y x_2 \\ x_1^2 x_2 - x_1 y \end{pmatrix}$$

Calculus

Example: $f(x_1, \ldots, x_d) = \|\mathbf{x}\|^2$

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Example: $f(x_1, \ldots, x_d) = \|\mathbf{x}\|^2 = \sum_{i=1}^{d} x_i^2$

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Let's see the details

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_1} x_i^2 = \frac{\partial}{\partial x_1} x_1^2 + \frac{\partial}{\partial x_1} x_2^2 + \ldots + \frac{\partial}{\partial x_1} x_d^2$$

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$$\nabla f(\mathbf{x}) = 2\mathbf{x}$$

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Reminder

A function $f : C \to \mathbb{R}$ over a convex set C is called *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

is satisfied for all $x, y \in C$ and $\lambda \in [0,1]$.

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This definition assumes any property of the function f

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A twice differentiable function $f: C \to \mathbb{R}$ over a convex set C is called *convex* if

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For a function of n variables the condition is on the Hessian which should be positive semi-definite

5. Verify that the function $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ with $g(x) := \frac{1}{2}x^2$ is convex.

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(c) Discuss the convex properties of the function $f(x) = ax^2$ where $a \in \mathbb{R}$.

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$$\frac{d}{dx}\left(-a\log x\right) = -\frac{a}{x}$$

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PROBABILITY & STATISTICS

Probability & statistics

Assume we have a random variable X with a finite no. of outcomes $x_1, x_2, \ldots, x_s$ and probabilities $\rho_1 = P(X = x_1), \rho_2 = P(X = x_2), \ldots, \rho_s = P(X = x_s)$. The expectation of X is defined as

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It is simply a weighted average!

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$$\implies \qquad \mathbb{E}_x[x_i] = \sum_{i=1}^{3} x_i \, \rho_i = \frac{1}{2} + \frac{11}{30} + \frac{1}{12} = \frac{19}{20} = 0.95$$

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Assume we have an absolutely continuous random variable X with probability density function ρ. The expectation of X is defined as

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Its square-root

$$\sigma_x := \sqrt{\text{Var}_x[x]}$$ is known as standard deviation

Probability & statistics

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The collection of random variables is independent and identically distributed (i.i.d.) if in addition we have

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Reminder

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Reminder

PDFs are defined in such a way that

$$\int_a^b \rho(x)dx = 1$$

For a uniform (and absolutely continuous) random variable X in $[0, 1]$ compute the expectation of $f(X)$ for

$$f(x) := \begin{cases} -\log(x) & x \in [0, 1/5] \\ 0 & \text{otherwise} \end{cases},$$

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(a) Consider the following probability density function of a (continuous) random variable x: $p(x|\alpha) = Ax^{-\alpha}$ for $x \geqslant 1$ where $A \in \mathbb{R}$ and $\alpha \in \mathbb{R}$. Compute the value A as function of α and discuss where it is defined.

Since the PDF need to be normalized and the variable is continuous we can write

$$\int_1^\infty p(x|\alpha)dx = 1$$

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In this case $A = 1 - \alpha$

(b) Compute the expectation value $\mathbb{E}[x]$ and the second moment of the distribution $\mathbb{E}[x^2]$ as function of α and discuss where they are defined.

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It can be easily seen how for any $\alpha \leqslant 2$ the expectation value is divergent. For $\alpha > 2$ we get $\mathbb{E}[x] = \frac{\alpha-1}{\alpha-2}$

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It can be easily seen how for any $\alpha \leqslant 2$ the expectation value is divergent. For $\alpha > 2$ we get $\mathbb{E}[x] = \frac{\alpha-1}{\alpha-2}$

(b) Compute the expectation value $\mathbb{E}[x]$ and the second moment of the distribution $\mathbb{E}[x^2]$ as function of α and discuss where they are defined.

By definition the second moment of the distribution reads

$$\begin{aligned} \mathbb{E}[x^2] &= \int x^2 p(x|\alpha)dx = (\alpha - 1)\int_1^{\infty} x^{2-\alpha} dx \\ &= (\alpha - 1)\frac{1}{3-\alpha} x^{3-\alpha}|_1^{\infty} \end{aligned}$$

It can be easily seen how for any $\alpha \leqslant 3$ the second moment is divergent. For $\alpha > 3$ we get $\mathbb{E}[x^2] = \frac{\alpha-1}{\alpha-3}$