PROBLEM SET 8 FOR MTH 6151

1. Verify that the following function expressed in polar coordinates are all harmonic functions, find their domain of definition (the region that the function is well defined), and then write down their expression in Cartesian coordinate.

(1) $\ln r$.
(2) θ.
(3) $r^2 \cos(2\theta)$.
(4) $r \ln r \sin\theta + r\theta\cos\theta$.

2. Suppose $U(x, y)$ is a harmonic function in $B_1(0)$ (disk of radius 1 centered at the origin) and $\lambda > 0$ is a constant, show that $V(x, y) = U(\lambda x, -\lambda y)$ is also a harmonic function, defined on the disk $B_{\frac{1}{\lambda}}(0)$.

3. (1) Find the harmonic function on the annular region

$$\Omega = \{\frac{1}{2} < r < 2\}$$

satisfying the boundary conditions given by

$$U(\frac{1}{2}, \theta) = 17 + 17\cos 2\theta + 17\sin 2\theta,$$
$$U(2, \theta) = 17 + 17\cos 2\theta + 17\sin 2\theta.$$

(2) Show that this is the unique harmonic function satisfying these boundary conditions.

4. Find the harmonic function on domain obtained by plane digged out a ball

$$\Omega = \{r \geq 1\} = \mathbb{R}^2 \setminus B_1(0)$$

satisfying the boundary conditions given by

$$U(1, \theta) = \cos\theta,$$

that satisfies $U(r, \theta) \to 0$ as $r \to \infty$.

5. Suppose that U is a harmonic function in the disk $\Omega = \{r < 2\}$ and that

$$U(2, \theta) = 3\sin\theta - 4\cos 4\theta + 1.$$

(1) Without finding the solution, compute the value of U at the origin —i.e. at $r = 0$.
(2) Without finding the solution, show that $U \geq -6$ and $U \leq 8$ on the whole disk Ω.

6. Use the maximum principle for the Laplace equation to show that the solution to the Poisson equation

$$\Delta U = f$$

over a domain Ω with boundary condition $U = 0$ on $\partial\Omega$ is unique.