WEEK 9 NOTES

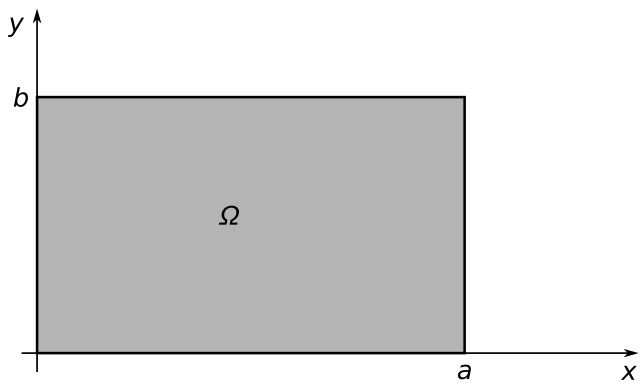
1. SEPARATION OF VARIABLES FOR THE LAPLACE EQUATION CONTINUED (READING MATERIAL, NOT TO BE TESTED)

Note. *We will leave this section's material of separation of variables of Laplace equations in Cartesian coordinates as a reading material. It will **not** be tested in the final exam!*

1.1. **Separation of variables in Cartesian coordinates.** The method of separation of variables can also be used in a symmetric domain in the (x, y) Cartesian coordinates. For example, consider the rectangular domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le a,\ 0 \le y \le b\}$$

as shown in the figure below:



We want to solve

$$\Delta U = 0, \qquad \text{on} \quad \Omega.$$

We make use of the method of separation of variables. To this end, given the geometric structure of the problem we look for solutions of the form

$$U(x, y) = X(x)Y(y).$$

Substituting the latter into

$$U_{xx} + U_{yy} = 0$$

one readily gets that

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Rearranging one finds that

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

The left hand side of this equality only depends on x while the left hand side only depends on y. Thus, both sides have to be equal to a constant. That is, one has

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k,$$

with k the *separation constant*. It follows then that one has the following ode's to solve:

$$X''(x) = kX(x), \tag{1.1a}$$

$$Y''(y) = -kY(y). \tag{1.1b}$$

The type of solutions depends on the sign of k. For example, if $k < 0$, then the solutions to (1.1a) are trigonometric functions while if $k > 0$ they are exponentials.

Note. *To determine k one needs* boundary conditions.

Example 1.1. Suppose that

$$U(0, y) = 0, \qquad U(a, y) = 0.$$

It follows from the above that

$$X(0) = X(a) = 0.$$

Accordingly, one needs to have $X(x)$ as periodic solutions —that is, one requires trigonometric functions and necessarily $k < 0$. We thus write

$$k = -\mu^2, \qquad \mu \quad \text{a constant.}$$

The solution to (1.1a) is then given by

$$X(x) = A\sin(\mu x) + B\cos(\mu x), \qquad A,\ B \quad \text{constants.}$$

Now, we have to match the boundary conditions. For this we observe that

$$\begin{aligned} &X(0) = B = 0, \\ &X(a) = A\sin(\mu a) + B\cos(\mu a) = A\sin(\mu a) = 0, \end{aligned}$$

where in the second line we have used that $B = 0$ (from the first line). Thus, in order to have $X(a) = 0$ one requires

$$\mu a = n\pi, \qquad n \in \mathbb{N}.$$

Hence, the required solution of (1.1a) is

$$X(x) = \sin\left(\frac{n\pi x}{a}\right).$$

We are now in the position of solving (1.1b) which takes the form

$$Y''(y) = \mu^2 Y(y).$$

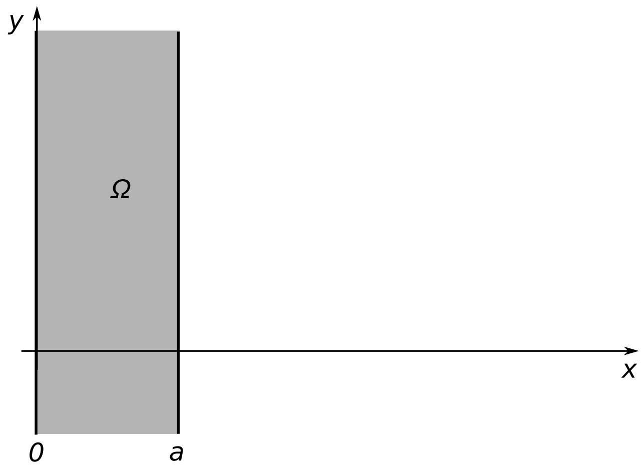
The solution can be expressed in terms of hyperbolic functions or, alternatively, trigonometric functions:

$$Y(y) = A_n \sinh\left(\frac{n\pi y}{a}\right) + B_n \cosh\left(\frac{n\pi y}{a}\right), \tag{1.2a}$$

$$Y(y) = C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}. \tag{1.2b}$$

The expression (1.2a) is conventionally used when the y-domain is finite while (1.2b) is used when it is infinite.

Note. *Observe that if one requires $U(x, y) \to 0$ as $y \to \infty$ one then necessarily has that $C_n = 0$ —see figure below.*



Putting everything together one ends up, for given n, with solutions of the form

$$\begin{aligned} U_n(x,y) &= \sin\left(\frac{n\pi x}{a}\right)\left(A_n \sinh\left(\frac{n\pi y}{a}\right) + B_n \cosh\left(\frac{n\pi y}{a}\right)\right), \\ &= \sin\left(\frac{n\pi x}{a}\right)\left(C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}\right). \end{aligned}$$

Now, recall that the equation $\Delta U = 0$ is linear —thus, the principle of superposition holds. The general solution is then a linear combination of all possible $U_n(x,y)$'s:

$$U(x,y) = \sum_{n=1}^{\infty} U_n(x,y).$$

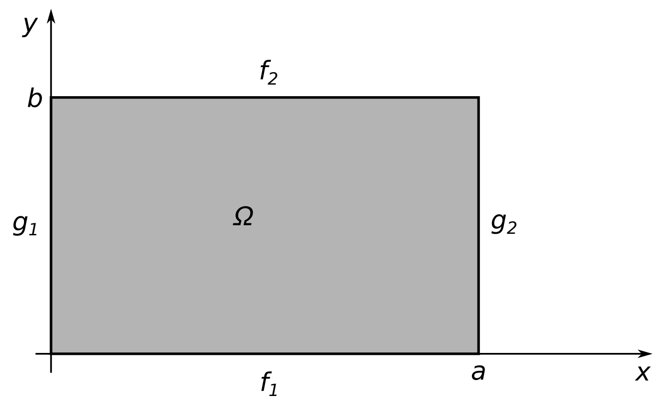
1.1.1. *The problem on a rectangle.* A more elaborated problem is:

$$U_{xx} + U_{yy} = 0,$$

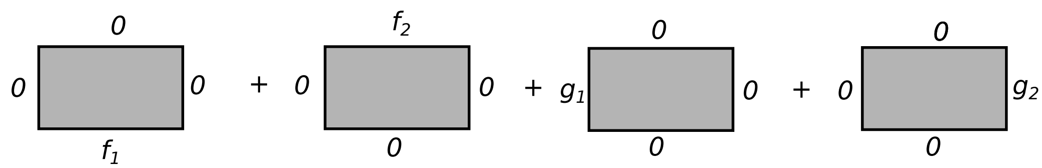
with boundary conditions

$$\begin{aligned} U(0,y) &= g_1(y), \\ U(a,y) &= g_2(y), \\ U(x,0) &= f_1(x), \\ U(x,b) &= f_2(x). \end{aligned}$$

A schematic depiction of the situation is given in the picture below:



Observe that the pde is homogeneous but the boundary conditions are not. To solve the problem above we exploit the linearity of the equation and the boundary conditions and break the original problem into 4 problems, each one with non-homogeneous boundary conditions:



In the following we concentrate, for conciseness on

$$\begin{aligned}
&\Delta U = 0,\\
&U(0,y) = U(a,y) = 0,\\
&U(x,0) = f_1(x),\\
&U(x,b) = 0.
\end{aligned}$$

From the discussion in the previous subsection we already know that

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \qquad n \in \mathbb{N}.$$

We also have that the general solution to $Y'' = \mu^2 Y$ is given by

$$Y(y) = B_n \cosh(\mu y) + A_n \sinh(\mu y).$$

However, one also needs that $Y(b) = 0$ so use the solution

$$Y(y) = B_n \cosh \mu(y-b) + A_n \sinh \mu(y-b), \tag{1.3}$$

which can be readily verified to solve $Y'' = \mu^2 Y$ (**exercise!**).

Note. *That* (1.3) *is also a solution to* $Y'' = \mu^2 Y$ *is, ultimately, a consequence of the fact that the Laplace equation is* translationally invariant.

The boundary condition $Y(b) = 0$ readily implies that $B_n = 0$. Hence, one has that

$$Y(y) = A_n \sinh \frac{n\pi}{a}(y-b).$$

Thus, the full solution for fixed $n \in \mathbb{N}$ is

$$U_n(x,y) = A_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi}{a}(y-b),$$

while the general solution is a sum of all the possibilities:

$$U(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi}{a}(y-b).$$

Finally, one needs to implement the boundary condition at $y = 0$. For this we observe that

$$\begin{aligned} U(x,0) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi}{a}(-b) \\ &= -\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{nb\pi}{a}\right) \\ &= f_1(x). \end{aligned}$$

Using the orthogonality property of the sine function one can find that

$$A_n = -\frac{\int_0^a f_1(x) \sin\left(\frac{n\pi x}{a}\right) dx}{\frac{a}{2} \sinh\left(\frac{nb\pi}{a}\right)}.$$

Let's apply this to the following example.

Example: Solve the following Dirichlet problem on the rectangle $\Omega\{(x,y)|0 \leq x \leq a, 0 \leq y \leq b\}$

$$\begin{aligned} &\Delta U = 0 \\ &U(x,0) = \sin \pi x - 2 \sin 2\pi x \\ &U(x,b) = 0 \\ &U(0,y) = 0 \\ &U(a,y) = 0. \end{aligned}$$

Using that $a = 2, b = \pi$ and the theory above, we have

$$\begin{aligned} A_2 &= -\frac{\int_0^a [\sin \pi x - 2 \sin 2\pi x] \sin \frac{2\pi x}{2} dx}{\sinh \frac{2\pi \cdot \pi}{2} + \frac{2}{2}} = \frac{-1}{\sinh(\pi^2)} \\ A_4 &= -\frac{\int_0^a [\sin \pi x - 2 \sin 2\pi x] \sin \frac{4\pi x}{2} dx}{\sinh \frac{4\pi \cdot \pi}{2} + \frac{2}{2}} = \frac{2}{\sinh(2\pi^2)} \\ A_n &= 0, \forall n \neq 2, 4. \end{aligned}$$

Thus

$$\begin{aligned} U(x,y) =& A_2 \sin \frac{2\pi x}{2} \sinh \frac{2\pi(y-\pi)}{2} + A_4 \sin \frac{4\pi x}{2} \sinh \frac{4\pi(y-\pi)}{2} \\ =& \frac{-1}{\sinh \pi^2} \sin \pi x \sinh[\pi(y-\pi)] + \frac{2}{\sinh(2\pi^2)} \sinh[2\pi(y-\pi).] \end{aligned}$$

One can also check the boundary condition boundary is satisfied as

$$\begin{aligned} U(x,0) =& \frac{-1}{\sinh \pi^2} \sin \pi x \sinh(-\pi^2) + \frac{2}{\sinh(2\pi^2)} \sinh(-2\pi^2) \\ =& \sin \pi x - 2 \sin 2\pi x. \end{aligned}$$

2. Invariant properties of harmonic functions

There are some invariant properties of the solutions to the Laplace equations in either polar or Cartesian coordinates.

In Cartesian coordinates, we have

Proposition 2.1. *If $U(x,y)$ is a harmonic function on a disk of radius r centerred at the origin, then $V(x,y) = U(\lambda x, \lambda y)$ and $W(x,y) = U(\lambda x, -\lambda y)$ are both harmonic functions on a disk of radius $\frac{r}{\lambda}$.*

In polar coordinates, we have

Proposition 2.2. *If $U(r,\theta)$ is a harmonic function on $\mathbb{R}^2 \setminus \{0\}$, then both $V(r,\theta) = U(\frac{1}{r}, \theta)$ and $W(r,\theta) = U(\frac{1}{r}, -\theta)$ are harmonic functions on $\mathbb{R}^2 \setminus \{0\}$.*

Both of these 2 properties are left as exercises in the problem sets/Courseworks.

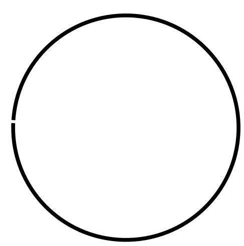
3. The mean value property

Several important properties of harmonic functions follow directly from Poisson's formula deduced at the end of last week.

$$U(r,\theta) = \frac{(r_*^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{f(\theta')d\theta'}{r_*^2 - 2rr_* \cos(\theta - \theta') + r^2}. \tag{3.1}$$

In particular, one has the following:

Proposition 3.1 (the first mean value property)**.** *Let U be a harmonic function on a disk Ω. Then the value of U at the centre of the disk is equal to the average of U on its circumference.*



Proof. Without loss of generality set the centre of the disk at the origin of the polar coordinates. Then, setting $r = 0$ in Poisson's formula (3.1) one obtains

$$U(0) = \frac{r_*^2}{2\pi} \int_0^{2\pi} \frac{f(\theta')}{r_*^2} d\theta'.$$

The latter can be rewritten as

$$U(0) = \frac{1}{2\pi r_*} \int_0^{2\pi} f(\theta') r_* d\theta'.$$

This is nothing but the average of $f(\theta)$ over the circumference —observe that $2\pi r_\star$ is the value of the circumference while $r_* d\theta'$ is the infinitesimal arc-length. □

Note. *The first mean value property allows one to determine the value of a harmonic function at the centre of the disk without actually having to solve the Laplace equation!*

Example 3.2. For the problem in Example 2.4 last week for a discontinuous temperature on the boundary, a quick calculation gives that

$$\begin{aligned} U(0) &= \frac{1}{2\pi}\int_0^{\pi} U_1 d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} U_2 d\theta \\ &= \frac{1}{2}(U_1+U_2). \end{aligned}$$

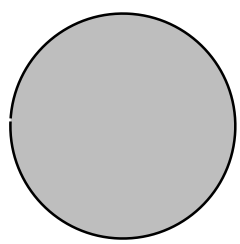
That is the value at the centre is the average of the two different (constant) values at the boundary —this is an intuitive observation.

Notice also that this coincide with the value we get when plugging in $r = 0$ in the solution obtained last week:

$$U(r,\theta) = \frac{U_1+U_2}{2} + \frac{U_1-U_2}{\pi}\sum_{m=1}^{\infty}\frac{r^m}{m}\big(1-(-1)^m\big)\sin m\theta.$$

There is a stronger version of the mean value property:

Proposition 3.3 (the second mean value property)**.** *Let U be a harmonic function on a disk Ω. Then the value of U at the centre of Ω equals the average on the disk.*



Proof. Let $r \le r_*$. The first mean value property then gives that

$$U(0) = \frac{1}{2\pi}\int_0^{2\pi} U(r,\theta)d\theta.$$

Multiplying by $2\pi r$ and integrating from 0 to r_* gives

$$\int_0^{r_*} 2\pi r U(0) dr = \int_0^{r_*}\int_0^{2\pi} U(r,\theta) r d\theta dr$$

However,

$$\int_0^{r_*} 2\pi r U(0) dr = 2\pi U(0)\int_0^{r_*} r dr = \pi r_*^2 U(0).$$

Hence,

$$U(0) = \frac{1}{\pi r_*^2}\int_0^{r_*}\int_0^{2\pi} U(r,\theta) r d\theta dr.$$

The above expression gives the value of u over the disk. In particular, πr_*^2 gives the area of the disk. □

Remark 3.4. One can write the second mean value property in the more concise form

$$U(0) = \frac{1}{\pi r_*^2} \int_{B_{r_*}(0)} U dV,$$

where $\mathcal{B}_{r_*}(0)$ denotes the ball (disk) of radius r_* centred at the origin.

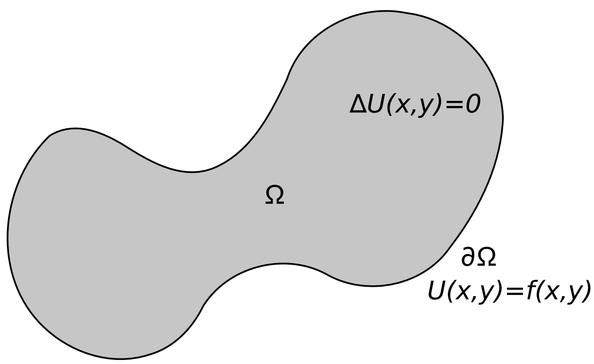
4. The maximum principle

In this section we will discuss the important properties of the maximum and minimum of harmonic functions. These properties have important application —mainly to discuss the uniqueness of solutions to the Laplace and Poisson equation.

We recall some *technical* concepts which will be used in the following discussion:

Open domain. An open domain (i.e. set) is one for which at every point in the set it is possible to have a sufficiently small ball (centred at the point in question) which is contained within the set. In particular, an open domain does not include its boundary.

Connected domain. A connected domain is one which consists only of one piece. More precisely, given two arbitrary points in a connected set, it is always possible to find a curve connecting the two points which is completely contained in the set.



The first result of this section is the following:

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and U be a harmonic function defined on Ω. Assume U achieves its maximum at a point $(x_\star, y_\star) \in \Omega$. Then $U(x,y)$ is constant for all $(x,y) \in \Omega$.*

Proof. Since $(x_\star, y_\star) \in \Omega$ and Ω is open, we can find $r > 0$ such that $\mathcal{B}_r(x_\star, y_\star) \subset \Omega$. By the mean value property we have that

$$U(x_\star, y_\star) = \frac{1}{\pi r^2} \int_{\mathcal{B}_r(x_\star, y_\star)} U(\underline{x}) d\underline{x}.$$

Since $U(x_\star, y_\star) \geq U(x,y)$ for all $(x,y) \in \Omega$ (it is a maximum!), then the only way to satisfy the mean value property is to have

$$U(x,y) = U(x_\star, y_\star) \qquad \text{for all } (x,y) \in \mathcal{B}_r(x_\star, y_\star).$$

Now, take any point $(x_n, y_n) \in \Omega$. We want to show that $U(x_n, y_n) = U(x_\star, y_\star)$. For this, we connect $(x_\star, y_\star)$ and (x_n, y_n) with a continuous curve that is covered by intersecting balls $\mathcal{B}_{r_0}(x_i, y_i)$, $2r_0 < r$, in such a way that

$$|(x_{i+1}, y_{i+1}) - (x_i, y_i)| < r_0, \qquad \text{for} \quad i = 0, \ldots, n-1.$$

By the first step in the proof one already knows that $U(x_1, y_1) = U(x_\star, y_\star)$. So, repeating the argument we obtain that

$$U(x_\star, y_\star) = U(x_i, y_i) \qquad \text{for} \quad i = 1, \ldots, n.$$

As the domain is connected then any point in it can be joined to $(x_\star, y_\star)$ by means of a curve completely contained in Ω. Thus, The argument used before shows that $U(x, y)$ must be constant throughout Ω. □

Changing U to $-U$ in the previous argument one obtains the following:

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and U be a harmonic function defined on Ω. Assume U achieves its minimum at a point $(x_\star, y_\star) \in \Omega$. Then $U(x, y)$ is constant for all $(x, y) \in \Omega$.*

Combining the above results one obtains the main result of this section:

Theorem 4.3 (the maximum/minimum principle)**.** *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain and U be a harmonic function defined on Ω. Then U attains its maximum and minimum values on the boundary $\partial\Omega$.*

Note. *In particular, if U is constant on $\partial\Omega$, then it is also constant on Ω.*

4.1. **Application to uniqueness.** The maximum principle is key to showing uniqueness of solution to the Laplace and Poisson equation.

Proposition 4.4. *Let $\Omega \subset \mathbb{R}^2$ be an open connected domain, then the Poisson equation on Ω*

$$\Delta U = \psi$$
$$U|_{\partial\Omega} = f,$$

has a unique solution

Proof. Suppose there are 2 solutions U_1, U_2, then $V = U_1 - U_2$ solves the following Laplace equation

$$\begin{cases} \Delta V = \Delta U_1 - \Delta U_2 = \psi - \psi = 0, \;\; \text{in } \Omega \\ V|_{\partial\Omega} = U_1|_{\partial\Omega} - U_2|_{\partial\Omega} = f - f = 0. \end{cases}$$

Then the maximum principle tells that $V \leq 0$ and $V \leq 0$. So $V \equiv 0$.

Namely $U_1 = U_2$ and the solution to the Poisson's equation is unique. □

Example 4.5. Suppose U is harmonic on the disk of radius 4 with boundary conditions $U(4, \theta) = 4 + 4\cos^2\theta$. Determine the maximum/minimum values of U in the disc of radius r and compute the value of U at the origin.

Notice $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$, we have $U(4, \theta) = 4 + 2\cos 2\theta + 2 = 6 + 2\cos 2\theta$.

By the mean value property

$$U(0) = \int_0^{2\pi} U(4, \theta) d\theta = 6.$$

By the maximum principle and the fact that $-1 \leq \cos\theta \leq 1$. we have

$$U_{max} = 6 + 2 = 8$$
$$U_{min} = 6 - 2 = 4.$$

Example 4.6. For a harmonic function on the annular region (see Problem set 8 Question 3)

$$\Omega = \{\frac{1}{2} < r < 2\}$$

satisfying the boundary conditions given by

$$U(\frac{1}{2}, \theta) = 17 + 17\cos 2\theta + 17\sin 2\theta,$$
$$U(2, \theta) = 17 + 17\cos 2\theta + 17\sin 2\theta.$$

Without solving it as in the problem sheets. We can show that $-17 \leq U \leq 51$ on the whole Ω.

Indeed, this follows from the maximum principle and the fact that $-1 \leq \cos 2\theta, \sin 2\theta \leq 1$.

$$U \geq 17 - 17 - 17 = -17$$
$$U \leq 17 + 17 + 17 = 51.$$

5. Basic concepts for the heat equation

In this chapter we study the $1+1$-dimensional heat equation —this is the paradigmatic example of parabolic equations:

$$U_t - \varkappa U_{xx} = 0,$$

with $\varkappa$ the so-called **diffusivity constant**. In $3+1$ dimensions the equation is given by

$$U_t - \varkappa\big(U_{xx} + U_{yy} + U_{zz}\big) = 0.$$

Thus, **time independent** solutions (i.e. with $U_t = 0$) satisfy the Laplace equation

$$\Delta U = 0.$$

We will interested in the following:

(i) **The boundary value problem**. Here one prescribes U at $t = 0$ and on $x = a$, $x = b$.
(ii) **The heat equation on the whole line.** In this case there are no boundary conditions and one only prescribes U at $t = 0$.

The heat equation has a wide range of applications in the study of heat propagation, diffusion of substances in a medium, finance, geometry...

5.1. General remarks.

Consider the $1+1$ heat equation in the form

$$U_t = \varkappa U_{xx}, \qquad \varkappa > 0. \tag{5.1}$$

Geometrically, given a function $U(x,t)$, the second derivative U_{xx} is the rate of change of slope (at fixed time) —that is, it determines whether the graph of U (for fixed t) is concave or convex. On the other hand, U_t is the rate of change of $U(x,t)$ at some fixed point. Thus, one has that

$U_t > 0$	if the graph of U(x,t) (for fixed t) is convex,
$U_t = 0$	if the graph is a straight line,
$U_t < 0$	if the graph is concave.

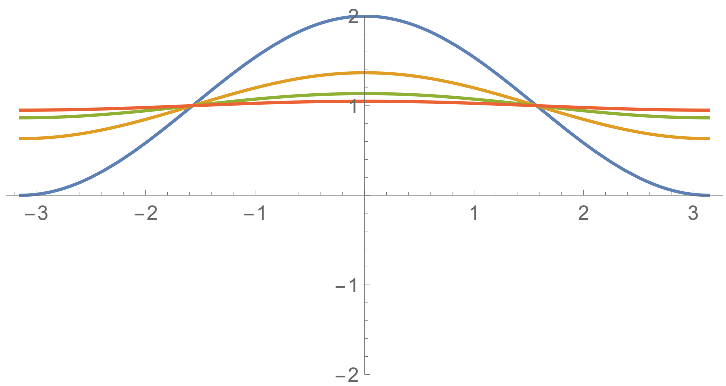
Thus, at all points x where $U_{xx} > 0$ we have that $U(x,t)$ increases in time , and at points where $U_{xx} < 0$ we have that $U(x,t)$ is decreasing in time.

Note. *The previous discussion shows that the effect of the heat equation is to* ***smooth out*** *bumps.*

Example 5.1. Consider the function

$$U(x,t) = 1 + e^{-\varkappa t} \cos x.$$

It can be checked to satisfy the heat equation. Plots of this function for various times are given below.



The plots show an initial central concentration spreading out an becoming more and more uniform as t increases. Observe, in particular how U increases where $U_{xx} > 0$ and decreases where $U_{xx} < 0$. Changing the value of $\varkappa$ affects the rate of smoothing: larger $\varkappa$ means faster smoothing and viceversa.

5.2. **Boundary conditions.** Recall that when solving first order ode's one needs one condition on the unknown (initial condition) to determine fully the solution. Since we want to predict the distribution of concentration/temperature $U(x,t)$ for all $t > 0$ and the heat equation has only one derivative in time, then at every x we need to prescribe one initial condition for $U(x,t)$ at $t = 0$ —that is

$$U(x,0) = f(x).$$

On the other hand, since $U_t = \varkappa U_{xx}$ contains U_{xx} and $x \in (a,b)$, we need to prescribe boundary conditions at the end points a and b at each time. This is consistent with the general principle for ode's that to solve second order boundary value problems one needs two boundary conditions (one at each point). These boundary conditions are determined by physical modelling and might contain U and U_x. The most common types are:

(i) **Dirichlet boundary conditions.** Here one prescribes

$$U(a,t) = h(t), \qquad U(b,t) = g(t).$$

These boundary conditions correspond to the temperature/concentration at the endpoints.

(ii) **Neumann boundary conditions.** Here one prescribes

$$U_x(a,t) = h(t), \qquad U_x(b,t) = g(t).$$

In this case one prescribes a flux of U rather than U itself. In particular, if

$$U_x(a,t) = U_x(b,t) = 0,$$

the endpoints are insulated —i.e. no flux.

(iii) **Mixed boundary conditions.** One can also have situations as

$$U_x(a,t) = h(t), \qquad U(b,t) = g(t),$$

or

$$U(a,t) = h(t), \qquad U_x(b,t) = g(t).$$

(iv) **Periodic boundary conditions.** One can also have

$$U(-a,t) = U(a,t)$$

or

$$U_x(-a,t) = U_x(a,t).$$

6. The heat equation on an interval

In this section we will see how the method of separation of variables can be used to obtain solutions to the heat equation on an interval. More precisely, we consider the following problem:

$$\begin{aligned}
&U_t = \varkappa U_{xx}, \qquad x \in [0,L], \quad t > 0, \\
&U(x,0) = f(x), \\
&U(0,t) = 0, \qquad U(L,t) = 0.
\end{aligned}$$

The boundary conditions describe, for example, a metallic wire whose ends are set (by means of some device) at a temperature of 0 degrees.

6.1. **Separation of variables.** Following the general strategy of the method we consider solutions of the form

$$U(x,t) = X(x)T(t).$$

Substitution into the heat equation gives

$$X(x)\dot{T}(t) = \varkappa X''(x)T(t).$$

Hence, dividing by XT we find that

$$\frac{\dot{T}(t)}{\varkappa T(t)} = \frac{X''(x)}{X(x)}.$$

We observe that the left hand side of this last expression only depends on x. The right hand side depends only on t. Thus, for the equality to hold one needs both sides to be constant. That is, one has that

$$\frac{\dot{T}(t)}{\varkappa T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

Thus, we end up with the following ordinary differential equations:

$$\dot{T} = -\varkappa - \lambda T, \tag{6.1a}$$

$$X'' = -\lambda X. \tag{6.1b}$$

Moreover, from the boundary conditions one has that

$$X(0)T(t) = 0, \qquad X(L)T(t) = 0,$$

so that

$$X(0) = X(L) = 0. \tag{6.2}$$

6.2. **Solving the equation for $X(x)$.** Combining equation (6.1b) with the boundary conditions (6.2) one obtains the eigenvalue problem

$$X'' = -\lambda X,$$
$$X(0) = X(L) = 0.$$

Notice we have already proved the following claim about eigenvalues when studying wave equations.

Claim 6.1. *The eigenvalues $\lambda \geq 0$.*

Thus, the general solution to equation (6.1b) is given by

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

Now, we make use of the boundary conditions. First we observe that

$$X(0) = A\cos 0 + B\sin 0.$$

Thus, from (6.2) it follows that

$$A = 0.$$

Using now $X(L) = 0$ one finds that

$$B\sin\lambda L = 0.$$

Clearly one needs $B \neq 0$ to get a non-trivial solution. Thus

$$\sqrt{\lambda} = \frac{\pi n}{L}, \qquad n = 1, 2, \ldots$$

Hence, the solution to the eigenvalue problem is given (ignoring the constant B) by

$$X_n(x) = \sin\left(\frac{\pi n x}{L}\right), \qquad \lambda_n = \frac{\pi^2 n^2}{L^2}.$$

6.3. **Solving the equation for $T(t)$.** Now knowing $\lambda = \frac{\pi^2 n^2}{L^2}$, we can solve for T

$$T_n(t) = Ce^{-\lambda_n \varkappa t} = Ce^{-\frac{\pi^2 n^2}{L^2}\varkappa t}, \qquad C \quad \text{a constant.}$$

6.4. **General solution.** The calculations from the previous sections can be combined to obtain the family of solutions to the heat equation

$$U_n(x,t) = X_n(x)T_n(t) = e^{-\frac{\pi^2 n^2}{L^2}\varkappa t}\sin\left(\frac{\pi n x}{L}\right).$$

The general solution is then applied using the principle of superposition:

$$U(x,t) = \sum_{n=1}^{\infty} a_n U_n = \sum_{n=1}^{\infty} a_n e^{-\frac{\pi^2 n^2}{L^2}\varkappa t}\sin\left(\frac{\pi n x}{L}\right), \tag{6.3}$$

with a_n constants that are fixed through the initial conditions.