WEEK 8 NOTES

1. BASIC IDEAS OF ELLIPTIC EQUATIONS

In this part of the course we will study the properties of elliptic equations in two dimensions (spatial). More precisely, we will look at the **Laplace equation**

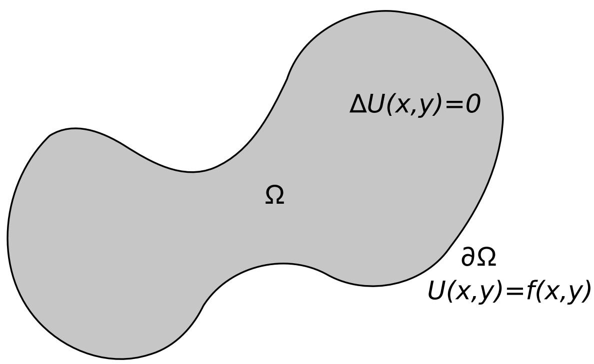
$$U_{xx} + U_{yy} = 0,$$

and the **Poisson equation**

$$U_{xx} + U_{yy} = f(x, y).$$

The Poisson equation is the inhomogeneous version of the Laplace equation.

Typically we will be interested in the so-called **Dirichlet problem** in which we solve the Laplace equation on a domain $\Omega \subset \mathbb{R}^2$ given that the value of U on the **boundary** $\partial\Omega$ of Ω is known.



Notation. *In what follows we write*

$$\Delta U = U_{xx} + U_{yy}.$$

The physicists notation is

$$\nabla^2 U = U_{xx} + U_{yy}.$$

*The operator Δ (∇^2) is called the **Laplacian**. The reason for the physicists notation is that the Laplacian is the divergence of the gradient of a function $\Delta U = \nabla \cdot \nabla U$.*

The Laplace and Poisson equations arise from applications in physics (electrostatics, Newtonian gravity), fluid flows (steady state), soap films, elastic membranes, and also in pure mathematics (complex variables). As examples consider the wave equation in $1 + 2$ dimensions

$$U_{tt} = c^2(U_{xx} + U_{yy})$$

and the $1 + 2$ heat equation

$$U_t = \varkappa(U_{xx} + U_{yy}).$$

For both of these equations it is of interest to look for solutions which are independent of time —i.e. $U_t = 0$. These solutions describe the **asymptotic behaviour** —i.e. at late times. This is a statement that is hard to show and that is at the forefront of modern pde research.

1.0.1. *Harmonic functions.*

Definition 1.1. A function having second partial derivatives on a domain $\Omega \subset \mathbb{R}^2$ is called harmonic if $\Delta U = 0$ for all $(x, y) \in \Omega$.

Example 1.2.

(i) the function $U(x, y) = x + y$ is harmonic for all $\Omega \subset \mathbb{R}^2$;
(ii) similarly for the function $U(x, y) = x^2 - y^2$;
(iii) the function $U(x, y) = \ln(x^2 + y^2)$ for any domain Ω not containing the origin as the function $U(x, y)$ is not defined there.

1.0.2. *Relation to complex variables.* Let $f(z) = u(x, y) + \mathrm{i}v(x, y)$ be an analytic function with $z = x + \mathrm{i}y$. To verify that the function $f(z)$ is analytic on a domain Ω one can make use of the *Cauchy-Riemann* equations:

$$v_y = u_x, \tag{1.1a}$$

$$v_x = -u_y. \tag{1.1b}$$

Applying $\partial/\partial y$ to equation (1.1a) one has that

$$v_{yy} = u_{xy} = -v_{xx}$$

where the second equality follows from (1.1b). Thus, one has that

$$v_{xx} + v_{yy} = 0,$$

that is, the imaginary part of analytic function is harmonic. A similar relation follows for the real part u.

Note. *This observation indicates a very deep connection between pde's and complex variables!*

2. Separation of variables for the Laplace equation

Before studying the general properties of the Laplace and Poisson equations, let us consider some explicit solutions using separation of variables.

2.1. Separation of variables in polar coordinates.

The method of separation of variables can be used to find solutions to the Laplace equation in settings with circular symmetry —i.e. a disk or an annulus.

Given the polar coordinates (r, θ) given by

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

the Laplacian can be expressed as

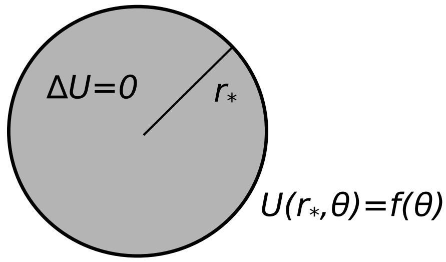
$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}.$$

Consider the boundary value problem for the Laplace equation in which the value of the solution is given on a circumference of a disk of radius r_* —namely,

$$\Delta U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0,$$

$$U(r_*, \theta) = f(\theta),$$

see the figure below:



Our task is to find the solution $U(r,\theta)$ in the interior of the circumference (**disk**). Following the general strategy of the method of separation of variables we look for solutions of the form

$$U(r,\theta) = R(r)\Theta(\theta).$$

Plugging into the Laplace equation in polar coordinates one obtains the expression

$$\Theta R'' + \frac{1}{r}\Theta R' + \frac{1}{r^2}R\Theta'' = 0.$$

Dividing the above expression by $R\Theta/r^2$ and rearranging one finds that

$$\frac{r^2R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta}.$$

The left hand side of the above expression depends only on r while the right hand side only on θ. Thus, each must be equal to some separation constant k —namely:

$$\frac{r^2R''}{R} + \frac{rR'}{R} = k, \qquad -\frac{\Theta''}{\Theta} = k,$$

or

$$\begin{aligned} &r^2R'' + rR' - kR = 0, \\ &\Theta'' + k\Theta = 0. \end{aligned}$$

The Θ-equation. This equation is used to set the value of k. Observe that we need periodic solutions so $k > 0$. In the following we write $k = m^2$. Then

$$\Theta(\theta) = A\cos m\theta + B\sin m\theta.$$

To enforce periodicity we require that

$$\begin{aligned} U(r,\theta) &= U(r,\theta+2\pi), \\ U_\theta(r,\theta) &= U_\theta(r,\theta+2\pi). \end{aligned}$$

Observing that

$$\cos m(\theta+2\pi) = \cos(m\theta + 2\pi m) = \cos m\theta$$

if $m \in \mathbb{N}$ (and similarly for $\sin m\theta$) then $m \in \mathbb{N}$.

The R-equation. Following the previous discussion one has that the equation for $R(r)$ takes the form

$$r^2R'' + rR' - m^2R = 0.$$

We look for solutions to this equations of the form

$$R(r) = r^\alpha,$$

for some constant α. It follows then that

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - m^2 r^\alpha = 0$$

so that

$$(\alpha^2 - m^2)r^\alpha = 0.$$

Hence, $\alpha^2 = m^2$ —that is,

$$\alpha = \pm m.$$

So the general solution for the R equation is

$$R(r) = C_m r^m + \frac{D_m}{r^m}.$$

For $m = 0$ one needs to do more work as there must be two independent solutions. In that case one has the equation

$$r^2 R'' + rR' = 0.$$

If $r \neq 0$ the latter implies

$$r\frac{dR'}{dr} = -R',$$

which can be read as an equation for R'. Integrating one obtains

$$R'(r) = \frac{D_0}{r},$$

from where a further integration gives

$$R(r) = C_0 + D_0 \ln r.$$

The general solution. Combining the whole of the previous discussion one finds that the general solution to the Laplace equation in polar coordinates is given by

$$U(r,\theta) = \Big(C_0 + D_0 \ln r\Big) + \sum_{m=1}^{\infty}\Big(C_m r^m + \frac{D_m}{r^m}\Big)(A_m \cos m\theta + B_m \sin m\theta). \tag{2.1}$$

2.2. **Examples of Laplace equations on disks and annuli.** Consider solutions such that $U(r_*,\theta) = f(\theta)$ and $U(r,\theta)$ well defined at the origin. Observe that the general solution as given by (2.1) are singular at $r = 0$. To avoid this behaviour set $D_0 = 0$ and $D_m = 0$. Hence,

$$U(r,\theta) = a_0 + \sum_{m=1}^{\infty} r^m (a_m \cos m\theta + b_m \sin m\theta), \tag{2.2}$$

where

$$a_m \equiv A_m C_m \qquad b_m \equiv B_m C_m.$$

The boundary condition at $r = r_*$ gives then

$$f(\theta) = a_0 + \sum_{m=1}^{\infty} r_*^m (a_m \cos m\theta + b_m \sin m\theta).$$

This is an example of a Fourier series! The Fourier coefficients can be computed (using the standard method) to be

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta,$$
$$a_n = \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta,$$
$$b_n = \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta.$$

Example 2.1. Let

$$\Delta U = 0, \text{ in } B_3(0)$$
$$U(3,\theta) = 2\cos^2\theta.$$

Namely $r^* = 3$ and $f(\theta) = 2\cos^2\theta = 1 + \cos 2\theta$.

We have $a_0 = 1$, $b_n = 0$ for all n and $a_n = 0$ except for $n = 2$. Moreover

$$a_2 = \frac{1}{\pi \cdot 3^2}\int_0^{2\pi} \cos^2(2\theta) = \frac{1}{9\pi}\int_0^{2\pi} \frac{\cos 4\theta + 1}{2} = \frac{1}{9}.$$

So $U(r,\theta) = 1 + \frac{1}{9}r^2\cos 2\theta$.

Alternatively, one can compute the coefficients by "observation". Notice that the boundary conditions gives

$$1 + \cos 2\theta = a_0 + \sum_{m=1}^{\infty} r_*^m(a_m \cos m\theta + b_m \sin m\theta)$$

and they have to be equal term by term.

The constant term agrees on both sides gives $a_0 = 1$.

And since the left hand side does not have all the $\cos m\theta$ and $\sin m\theta$ terms except for $\cos 2\theta$ with $m = 2$. So all the a_m and b_m are vanishing except for $m = 2$.

And the coefficients in front of $\cos 2\theta$ agrees, $1 = a_2 \cdot 3^2$. So $a_2 = \frac{1}{9}$.

This gives the same conclusion!

Example 2.2. Let $r_* = 1$ and $f(\theta) = \sin\theta$. One can then evaluate equation (2.2) at $r = 1$to yield

$$U(1,\theta) = a_0 + \sum_{m=1}^{\infty} a_m \cos m\theta + \sum_{m=1}^{\infty} b_m \sin m\theta.$$

As, on the other hand,

$$U(1,\theta) = \sin\theta$$

and the sines and cosines are independent, then by direct inspection one finds that

$$a_0 = 0, \qquad a_m = 0,$$
$$b_1 = 1, \qquad b_m = 0, \quad m \neq 1.$$

So, in this case, the solution takes the simple form

$$U(r,\theta) = r\sin\theta.$$

Example 2.3. Let $r_* = 1$ and $f(\theta) = \cos^2\theta$. Recall the identity

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta.$$

So, in this case we have that

$$\frac{1}{2} + \frac{1}{2}\cos 2\theta = a_0 + \sum_{m=1}^{\infty} a_m \cos m\theta + \sum_{m=1}^{\infty} b_m \sin m\theta,$$

from where direct inspection yields

$$\begin{aligned} &a_0 = \frac{1}{2}, \qquad a_2 = \frac{1}{2}, \qquad a_m = 0, \quad m \neq 0,\, 2, \\ &b_m = 0. \end{aligned}$$

Thus, the solution is given by

$$U(r,\theta) = \frac{1}{2} + \frac{1}{2} r^2 \cos 2\theta.$$

Example 2.4. Now, suppose that the boundary conditions are such that on half of the circle the function takes the constant value U_1 and in the lower part it takes the value U_2. More precisely, one has that

$$f(\theta) = \begin{cases} U_1 & 0 < \theta < \pi \\ U_2 & \pi < \theta < 2\pi. \end{cases}$$

Assume, further for simplicity that $r_* = 1$.

From the theory of Fourier series we have that

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta) d\theta = \frac{U_1}{2\pi}\int_0^{\pi} d\theta + \frac{U_2}{2\pi}\int_{\pi}^{2\pi} d\theta = \frac{U_1 + U_2}{2} \\ a_m &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos m\theta d\theta = \frac{U_1}{\pi}\int_0^{\pi} \cos m\theta d\theta + \frac{U_2}{\pi}\int_{\pi}^{2\pi} \cos m\theta d\theta = 0. \end{aligned}$$

However, one also has that

$$\begin{aligned} b_m &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin m\theta d\theta = \frac{U_1}{\pi}\int_0^{\pi} \sin m\theta d\theta + \frac{U_2}{\pi}\int_{\pi}^{2\pi} \sin m\theta d\theta \\ &= -\frac{U_1}{\pi m}\big((-1)^m - 1\big) - \frac{U_2}{\pi m}\big(1 - (-1)^m\big) = \frac{(U_1 - U_2)\big(1 - (-1)^m\big)}{\pi m}. \end{aligned}$$

Hence, the solution to the Laplace equation is given by

$$U(r,\theta) = \frac{U_1 + U_2}{2} + \frac{U_1 - U_2}{\pi}\sum_{m=1}^{\infty} \frac{r^m}{m}\big(1 - (-1)^m\big)\sin m\theta.$$

Observe that the solution only contains terms with m odd. In Exercise 6 of Coursework 7, you will be asked to write the above series in closed form.

Consider now $\Delta U = 0$ in $\Omega = \{r_1 \leq r \leq r_2\}$ —this type of region is called an annulus (ring) with inner radius r_1 and outer radius r_2. Boundary conditions are then given by

$$\begin{aligned} U(r_1,\theta) &= f(\theta), \\ U(r_2,\theta) &= g(\theta). \end{aligned}$$

In this case one can keep the general solution as the origin is excluded.

Example 2.5. Suppose $\Delta U = 0$ in $\Omega = \{1 \leq r \leq e\}$ and

$$\begin{aligned} U(1,\theta) &= 4 + (2 + e^2)\sin 2\theta) \\ U(e,\theta) &= (2e^2 + 1)\sin 2\theta. \end{aligned}$$

Recall the general solution is of the form

$$U(r,\theta) = \Big(C_0 + D_0 \ln r\Big) + \sum_{m=1}^{\infty}\Big(C_m r^m + \frac{D_m}{r^m}\Big)(A_m \cos m\theta + B_m \sin m\theta).$$

We must have:

The constant terms satisfies

$$4 = C_0 + D_0 \ln 1$$
$$0 = C_0 + D_0 \ln e.$$

Thus solving it we get $C_0 = 4$ and $D_0 = -4$.

Next, notice that there are only $\sin$ terms and no $\cos$ terms, we have all $A_m = 0$ for all m.

And since there is only $\sin(2\theta)$ terms, we must have all $B_m = 0$ for $m \neq 2$.

Moreover, we can assume $B_2 = 1$ and solve for C_2, D_2. The coefficients in front of the $\sin 2\theta$ terms gives

$$2 + e^2 = C_2 \cdot 1^2 + \frac{D_2}{1^2}$$
$$2e^2 + 1 = C_2 \cdot e^2 + \frac{D_2}{e^2}.$$

Solve it we get $C_2 = 2$ and $D_2 = e^2$.

So the solution is

$$U(r,\theta) = 4 - 4\ln r + (2r^2 + \frac{e^2}{r^2})\sin(2\theta).$$

2.3. **Poisson's formula.** We have obtained previously the solution to Dirichlet's problem on a disk in the form of the infinite series

$$U(r,\theta) = a_0 + \sum_{m=1}^{\infty} r_*^m (a_m \cos m\theta + b_m \sin m\theta),$$

with

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta,$$
$$a_n = \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta,$$
$$b_n = \frac{1}{\pi r_*^n}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta.$$

Remarkably, the previous solution can be written in closed form —i.e. in a way it does not involve an infinite series.

2.3.1. *Some useful facts.* We recall some useful fact that will be used in the following calculation.

Writing trigonometric functions in terms of exponentials. One has the Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where $i = \sqrt{-1}$. From the above expression it follows that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Geometric series. Recall that for $|x|<1$ one has that

$$1+x+x^2+\cdots=\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}.$$

So, in particular

$$x+x^2+\cdots=\sum_{n=1}^{\infty}x^n=\frac{x}{1-x}. \tag{2.3}$$

2.3.2. *Expressing the series solution in closed form.* Substituting the expressions for the Fourier coefficients into the general solutions of Laplace equations on the disk (2.2) one obtains

$$\begin{aligned}U(r,\theta)&=\frac{1}{2\pi}\int_0^{2\pi}f(\theta')d\theta'+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{r^m}{r_*^m}\left(\int_0^{2\pi}f(\theta')\cos m\theta' d\theta'\cos m\theta+\int_0^{2\pi}f(\theta')\sin m\theta' d\theta'\sin m\theta\right)\\&=\frac{1}{2\pi}\int_0^{2\pi}f(\theta')d\theta'+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{r^m}{r_*^m}\int_0^{2\pi}f(\theta')\big(\cos m\theta'\cos m\theta+\sin m\theta'\sin m\theta\big)d\theta'.\end{aligned}$$

Recalling that

$$\cos m(\theta-\theta')=\cos m\theta'\cos m\theta+\sin m\theta'\sin m\theta,$$

one gets then

$$\begin{aligned}U(r,\theta)&=\frac{1}{2\pi}\int_0^{2\pi}f(\theta')d\theta'+\frac{1}{\pi}\int_0^{2\pi}f(\theta')\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m\cos m(\theta-\theta')d\theta'\\&=\frac{1}{2\pi}\int_0^{2\pi}f(\theta')\left(1+2\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m\cos m(\theta-\theta')\right)d\theta'.\end{aligned}$$

Now, rewriting $\cos m(\theta-\theta')$ in terms of integrals one finds that

$$\begin{aligned}1+2\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m\cos m(\theta-\theta')&=1+\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m\left(e^{im(\theta-\theta')}+e^{-im(\theta-\theta')}\right)\\&=1+\sum_{m=1}^{\infty}\left(\frac{re^{i(\theta-\theta')}}{r_*}\right)^m+\sum_{m=1}^{\infty}\left(\frac{re^{-i(\theta-\theta')}}{r_*}\right)^m.\end{aligned}$$

The last two terms are geometric series like in (2.3) series with x given by the expressions in brackets. Accordingly, we can write

$$\begin{aligned}1+2\sum_{m=1}^{\infty}\left(\frac{r}{r_*}\right)^m\cos m(\theta-\theta')&=1+\frac{(r/r_*)e^{i(\theta-\theta')}}{1-(r/r_*)e^{i(\theta-\theta')}}+\frac{(r/r_*)e^{-i(\theta-\theta')}}{1-(r/r_*)e^{-i(\theta-\theta')}}\\&=1+\frac{re^{i(\theta-\theta')}}{r_*-re^{i(\theta-\theta')}}+\frac{re^{-i(\theta-\theta')}}{r_*-re^{-i(\theta-\theta')}}\\&=\frac{r_*^2-r^2}{r_*^2-2rr_*\cos(\theta-\theta')+r^2}.\end{aligned}$$

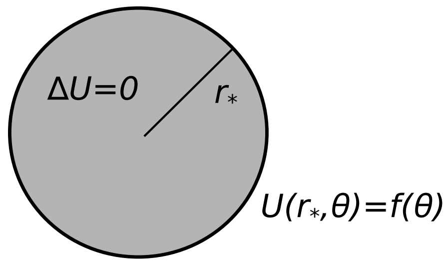
Thus, one has that

$$U(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\theta')(r_*^2-r^2)}{r_*^2-2rr_*\cos(\theta-\theta')+r^2}d\theta',$$

or after some rearrangements

$$U(r,\theta) = \frac{(r_*^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{f(\theta')d\theta'}{r_*^2 - 2rr_*\cos(\theta-\theta') + r^2}. \tag{2.4}$$

The latter is known as *Poisson's formula*. It expresses the solution to the Dirichlet problem on a disk as an integral of the boundary data over the boundary of the disk.



Note. *The term*

$$r_*^2 - 2rr_*\cos(\theta-\theta') + r^2$$

is, essentially the cosine's law *of trigonometry and gives the distance between a point with polar coordinates* (r,θ) *in the interior of the disk where we want to know the value of* U *and the points* (r_*,θ') *on the boundary of the disk (over which one is integrating).*